ON THE CONNECTIVITY OF COOPERATIVE AND NON COOPERATIVE WIRELESS COMMUNICATION NETWORKS

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ABSTRACT

In a seminal work, Gupta and Kumar derived the conditions for the asymptotic connectivity of a network composed of nodes uniformly distributed over a disc of unit area, as the number of nodes goes to infinity [1]. In this work, we incorporate the channel fading and we provide the conditions for the network connectivity, in case of single or multi-antenna transceivers. In particular, we derive closed form, albeit approximate, expressions for the spatial density with which the nodes must be deployed in order to insure the network connectivity with a desired probability. Finally, we show how to improve the connectivity by allowing nearby nodes to transmit in a cooperative manner, using a distributed space-time coding strategy, in order to get spatial diversity gain.

1. INTRODUCTION

One of the fundamental issues in the design of a wireless network is the minimal transmit power that guarantees the network connectivity, i.e. the property that each node is connected to each other node. In a seminal work, Gilbert modelled the network topology as a homogeneous two-dimensional Poisson point process, with constant density \( \lambda \), where two points are linked if their distance is less than a coverage radius \( r(n) \) that depends, on its turn, on the transmit power. Starting from this simple, yet meaningful, model and applying, for the first time, basic concepts from continuum percolation, Gilbert showed that there exists a critical value \( \lambda_c \) of the node density such that, for \( \lambda < \lambda_c \), the network is disconnected whereas for \( \lambda > \lambda_c \) there is a nonnull probability that the network contains a giant component composed of an infinite number of nodes, thus creating the possibility for long distance communications through multiple hops [2]. Of course, the existence of a component with infinite nodes does not guarantee the network connectivity, as some nodes could be isolated from that giant component. The connection between continuum percolation, covering strategies and the geometry of wireless networks has been further investigated quite recently in [3].

In another fundamental paper, Gupta and Kumar derived the minimum (critical) power that guarantees the network connectivity with probability one [1]. More specifically, Gupta and Kumar proved that if the network is composed of a set of \( n \) nodes randomly distributed over a disc of unit area and each node transmits with a power that allows the coverage of a circle of radius \( r(n) \) that scales with \( n \) as follows

\[
r = \sqrt{\frac{\log n + c(n)}{n}},
\]

the network is asymptotically connected with probability one if and only if \( c(n) \) tends to infinity as \( n \) goes to infinity.

In this work we assume, as in Gilbert’s work, a homogeneous Poisson point process and we find the coverage radius \( r_0(n) \) that guarantees that there are no isolated nodes, where the connection between nodes is defined in terms of out-of-service probability, with respect to a target BER, for a given channel fading model. Then, exploiting Penrose’s theorem [4] on the relationship between the connectivity and the degree of a random geometric graph, we derive asymptotic results on the network connectivity. Building on these derivations, we show how the connectivity improves by using multiantenna terminals. Finally, we extend the analysis to cooperative networks where nearby nodes, i.e. nodes within the coverage of each other, transmit in a coordinated manner according to a distributed space-time coding strategy, in order to achieve transmit diversity gain.

2. CONNECTIVITY OF A RANDOM GEOMETRIC GRAPH

We start reviewing some results on the connectivity of geometric random graphs, as derived by Bettstetter in [5]. We assume that the nodes are distributed according to a two dimensional (2D) Poisson point process, with constant spatial density \( \rho \). One of the key properties of this kind of point process is that the pdf of the distance between a point and its nearest neighbor is a Rayleigh pdf. In formulas, denoting with \( \Xi \) the random variable indicating the distance between one node and its nearest neighbor, the pdf of \( \Xi \) is

\[
p_p(\xi) = 2\pi \rho \xi e^{-\rho \xi^2},
\]

where \( \rho \) is density of the Poisson process. Starting from (2), and denoting by \( r_0(n) \) the coverage radius of each node, the probability that a node is isolated is

\[
P_{\text{isolated}} = 1 - P\{\xi \leq r_0(n)\} = e^{-\pi \rho r_0^2(n)}.
\]

The probability that there are no isolated nodes or that, equivalently, the degree of the graph is strictly greater than zero, is, approximately [5]

\[
P_{\text{con}} \simeq (1 - e^{-\pi \rho r_0^2(n)})^n.
\]

This expression is approximated as it implicitly assumes that the events for which the nodes are not isolated are statistically independent of each other. Nevertheless, the validity of this approximation was tested in [5] and we will also test it through simulation results later on. Instead, the validity of the approximation
depends on the product $\rho r_m^2(n)$. If $\rho r_m^2(n) \ll 1$, the approximation is clearly not valid at all. Conversely, if $\rho r_m^2(n) \gg 1$, the approximation is more and more valid. On the other hand, since we are interested in the event that the network is connected with high probability, i.e. $p_{\text{conn}}$ is close to one, this implies that we are interested in the case where the product $\rho r_m^2(n)$ is high, and then we can use the approximation. Within the validity of the approximation leading to (4), inverting (4), the coverage radius $r_0$ that ensures that the degree of the network is greater than zero is:

$$r_0(n) = \sqrt{-\log \left(1 - \frac{1}{p_{\text{conn}}(n)}\right) \pi \rho}.$$  

(5)

Of course, this radius does not guarantee the network connectivity, as the network could be composed of isolated clusters, with no isolated nodes. In general, in fact, the radius that guarantees the global connectivity is greater than the radius that simply guarantees that the degree of the network is strictly greater than zero. Nevertheless, Penrose proved that there is a basic relationship between connectivity and degree of a geometric graph [4]. In particular, Penrose proved that, denoting with $r(\kappa \geq k)$ the minimum distance $r_0$ such that a random graph is $k$-connected and with $r(\delta \geq k)$ the minimum distance such that the graph has minimum degree $k$, then [4]

$$\lim_{n \to \infty} \Pr\{r(\kappa \geq k) = r(\delta \geq k)\} = 1, \quad \forall k.$$  

(6)

This means that, asymptotically as $n$ tends to infinity, the coverage radius that insures a degree $k$ tends to coincide with the radius that yields connectivity $k$. Hence, the equivalence (6) guarantees that, asymptotically, as $n$ tends to infinity, the radius $r_0(n)$ in (5) tends to coincide with the radius that guarantees also the connectivity. Hence, the value $r_0(n)$ in (5) denotes, approximately, the minimum coverage that each node has to provide in order to guarantee the connectivity of the whole graph, with a given probability.

At a first glance, it might appear that the coverage radius $r_0(n)$ in (5) contradicts Gupta and Kumar result, recalled in (1). Conversely, we show next that (5) and (1) are in good agreement. In fact, if we rewrite the connectivity probability $p_{\text{conn}}(n)$ as $p_{\text{conn}}(n) = 1 - \epsilon(n)$, with $\epsilon(n) \ll 1$, we may use the first order Taylor series expansion of $p_{\text{conn}}(n)^{1/\gamma}$ as $p_{\text{conn}}(n)^{1/\gamma} \approx 1 - \epsilon(n)/n$. As a consequence, (5) becomes, approximately,

$$r_0(n) \approx \sqrt{\log n - \log \epsilon(n) / \pi \rho}.$$  

(7)

Since in Gupta and Kumar the node density is $\rho = n$, inserting this value in (7), we can verify that (1) is in perfect agreement with (5) if we set $c(n) = -\log(\epsilon(n))$. In fact, requiring asymptotic connectivity with probability one implies that $\lim n \to \infty \epsilon(n) = 0$ and this corresponds to the condition $\lim n \to \infty c(n) = \infty$ found in [1]. Interestingly, the equivalence between these two different derivations gives also a meaning to the term $c(n)$ appearing in [1].

A test of validity of the approximations leading to (4) is reported in Fig. 1, that shows $p_{\text{conn}}$ as a function of the SNR (referred to the transmit side), defined as $SNR_T := pr_T/N_0$, as given in (4) (solid line), and the probability that the network is connected (dashed line), estimated over 200 independent realizations of random geometric graphs composed of 1000 points distributed over a toroidal surface, to avoid border effects. We can see a very good agreement between theory and simulation. The slight difference is due to the fact that the independence assumption is not exactly valid and that (6) is valid only for $n$ going to infinity. It is interesting to notice, from Figure 1, the rapid change from probability zero to probability one. This is indeed a characteristic of random geometric graphs, that is reminiscent of phase transitions in chemistry.

If we wish to increase the fault tolerance of the graph, we need to increase the connectivity order. It was shown in [5] that the probability that a graph is $k$-connected is approximately equal to the probability that the minimum degree of the graph is at least $k$; that is

$$Pr_k := \Pr\{G \text{ is } k\text{-connected}\} \approx \Pr\{d_{\min} \geq k\} = \left[1 - \sum_{i=0}^{k-1} \left(\frac{\rho \pi r_0^2}{4}ight)^i e^{-\rho \pi r_0^2}\right]^n.$$  

(8)

This expression does not admit an inverse in closed form, but it is certainly invertible as it is a monotonic increasing function of $r_0$. The value of $r_0$ providing the desired value $Pr_k$, to be found numerically, guarantees the $k$-connectivity, within the limits of approximations in (8).

3. CONNECTIVITY OF A WIRELESS NETWORK

In a wireless network, besides the position of the radio nodes, there is one more source of randomness: the fading of the radio links. The connectivity of a wireless network is then, in general, much more difficult than the connectivity of a random graph. An interesting recent approach is proposed in [6] (see also the references therein). In this paper, we propose an alternative definition of connectivity of a wireless network. In case of fading, it is in fact necessary to define first the concept of connectivity. We introduce the out-of-service probability $P_{\text{out}}$, defined as the probability that the bit error rate (BER) exceeds a target BER, let us say $P_0$, and
we say that two nodes are linked to each other if the out-of-service probability on their link does not exceed the prescribed value $P_{\text{out}}$. This is equivalent to say that the BER on the link may exceed the target BER for no more than a percentage $P_{\text{out}}$ of time. In the following sections, we show how to choose the coverage radius in order to accommodate for such a system requirement. To simplify the theoretical analysis, we assume that the out-of-service event occurs with a probability smaller than a given variance $\sigma^2$, and that all links are interference-free. This means that the system is implicitly allocating orthogonal channels to different links, with a consequent rate loss.

### 3.1. Connectivity of a SISO Flat-Fading Network

Using QAM constellations, the bit error rate is

$$P_e = c_M \cdot Q \left( \sqrt{\frac{g_M}{N_0}} |h|^2 \right) \leq c_M \cdot e^{-g_M \frac{|h|^2}{N_0}}. \quad (9)$$

where $E_b$ is the energy per bit, $N_0$ is the noise variance, $h$ is the flat-fading coefficient, and $c_M$ and $g_M$ are two coefficients that depend on the order $M$ of the QAM constellation as follows [9]

$$c_M = \frac{4}{\sqrt{M} - 1} \cdot \log_2 M,$$

$$g_M = \frac{3}{\sqrt{M} - 1} \cdot \log_2 M. \quad (10)$$

We assume that the channels are Rayleigh fading, so that $|h|^2$ is an exponential random variable with expected value $\sigma^2 = 1/r^\alpha$, where $r$ is the link length and the exponent $\alpha$ depends on the environment where the propagation takes place. Typically, $\alpha$ is between two and six. Exploiting the upper bound in (9), we can upper-bound the out-of-service probability as

$$P_{\text{out}} = \Pr\{P_e > P_0\} \leq \Pr\{c_M \cdot e^{-g_M \frac{|h|^2}{N_0}} > P_0\}. \quad (11)$$

Hence $P_{\text{out}}$ can be upper-bounded as

$$P_{\text{out}} \leq 1 - e^{-N_0 \log(c_M/P_0)/(g_M \cdot \sigma^2)}. \quad (12)$$

We say that a link is reliable, and it is then established, if the out-of-service event occurs with a probability smaller than a given value. Setting $\sigma^2 = 1/r^\alpha$ in (12) and inverting (12), we find the coverage radius

$$r_{\text{cov}} = \left[ \frac{g_M \cdot \sigma^2 \cdot \log(1 - P_{\text{out}})}{N_0 \cdot \log(c_M/P_0)} \right]^{1/\alpha}. \quad (13)$$

Since $P_{\text{out}}$ is typically small, we can use the approximation $\log(1 - P_{\text{out}})$ to rewrite (13) as

$$r_{\text{cov}} \approx \left[ \frac{g_M \cdot \sigma^2 \cdot P_{\text{out}}}{N_0 \cdot \log(c_M/P_0)} \right]^{1/\alpha}. \quad (14)$$

Equating (14) to (5), we get the relationship between the node density and the transmitted power necessary to insure the network connectivity, for a given number of nodes $n$. For example, the minimum SNR that guarantees the network connectivity, with probability $p_{\text{con}}$, for a given node density $\rho$, is:

$$\frac{\sigma^2}{N_0} \approx \frac{\log(c_M/P_0)}{g_M \cdot P_{\text{out}}} \cdot \frac{-\log(1 - \frac{1}{p_{\text{con}}})}{\pi \rho}^{\alpha/2}. \quad (15)$$

As expected, we see that to achieve a desired probability of connectivity, we can decrease the transmitted energy if we increase the node density $\rho$. At the same time, the transmitted energy must increase when $\alpha$ increases. In a sensor network scenario, where it is important to foresee, a priori, the density with which the nodes should be deployed, in a given area, in order to guarantee a certain probability of connectivity, (15) can be rewritten by making explicit the minimum node density $\rho$ that guarantees the connectivity, for a given set of sensors having a specified transmitted power

$$\rho_{\text{min}} = -\frac{\log(1 - p_{\text{con}})}{\pi} \cdot \frac{1}{g_M \cdot P_{\text{out}}} \left( \frac{N_0 \cdot \log(c_M/P_0)}{E_b} \right)^{2/\alpha}. \quad (16)$$

### 3.2. Connectivity of a MIMO Flat-Fading Network

We show now how the connectivity improves if the radio nodes have multiple antennas. In such a case, the network may benefit from the diversity gain. To make a fair comparison with the single antenna case seen before, denoting with $n_T$ the number of transmit/receive antennas, we set the power transmitted by each antenna equal to $p_T/n_T$, where $p_T$ is the overall transmit power.

Using an $n_T \times n_T$ MIMO system, using a space-time coding technique capable of achieving full diversity, the error probability is

$$P_e = c_M \cdot Q \left( \sqrt{\frac{g_M \cdot \sigma^2 \cdot \log(c_M/P_0)}{N_0 \cdot n_T}} \right) \leq c_M \cdot e^{-g_M \frac{|h|^2}{N_0 \cdot n_T}}, \quad (17)$$

having introduced, in the last approximation, the random variable $z := \sum_{i=1}^{n_T} |h_i|^2$. The out-of-service probability can then be upper bounded as follows

$$P_{\text{out}} = D_Z \left( \frac{N_0 \cdot n_T \cdot \log(c_M/P_0)}{g_M \cdot \sigma^2} \right).$$

where $D_Z(z)$ is the cumulative distribution function (CDF) of $z$. If the channels are Rayleigh fading, independent, and all with the same variance $\sigma^2$, the CDF of $z$ is

$$D_Z(z) = 1 - e^{-z/\sigma^2} \cdot \sum_{k=0}^{n_T^2 - 1} \left( \frac{z}{\sigma^2} \right)^k \cdot \frac{1}{k!}. \quad (18)$$

For small values of $z$, more specifically for $z \ll \sigma^2$, $D_Z(z)$ can be approximated as

$$D_Z(z) \approx \left( \frac{z}{\sigma^2} \right)^{n_T^2} \cdot \frac{1}{n_T^2 \cdot \pi}. \quad (19)$$

For the sake of finding closed form, albeit approximated, expressions, it is useful to introduce the normalized random variable $x = z/\sigma^2$, whose CDF is

$$D_X(x) = 1 - e^{-x} \cdot \sum_{k=0}^{n_T^2 - 1} x^k \cdot \frac{1}{k!} \approx x^{n_T^2} \cdot \frac{1}{n_T^2 \cdot \pi}. \quad (20)$$

\footnote{We must keep in mind, however, that this result has been obtained by assuming lack of interference.}
Repeating the same kind of derivations as in the SISO case, the out-of-service probability for the MIMO case can be written as

\[ P_{\text{out}} \leq D_X \left( \frac{N_0 n_T \log(c_M/P_0)}{g_M \mathcal{E}_b \sigma_k^2} \right). \]  

(21)

From (21), setting \( \sigma_k^2 = 1/r^\alpha \), we can derive the coverage of each node

\[ r_{\text{cov}} = \left[ \frac{g_M \mathcal{E}_b}{N_0 n_T \log(c_M/P_0)} \right] D_X^{-1}(P_{\text{out}})^{1/\alpha}. \]  

(22)

To derive an approximate closed form expression for \( r_{\text{cov}} \), since we are interested in small values of the out-of-service probability, we can use the approximation (20) to invert \( D_X(x) \). The result is

\[ r_{\text{cov}} \simeq \left[ \frac{g_M \mathcal{E}_b}{N_0 n_T \log(c_M/P_0)} \left( n_T^2 P_{\text{out}} \right)^{1/\alpha-2} \right]^{1/\alpha}. \]  

(23)

Equating (22) to (5), we get the minimum transmit power guaranteed

\[ \frac{\mathcal{E}_b}{N_0} = n_T \log(c_M/P_0) \left( \frac{1 - p_{\text{con}}}{\pi} \right)^{\alpha/2} \]  

(24)

As a numerical example, in Figure 2 we show the density \( \rho \) as a function of the transmitted energy per bit, normalized to the noise power, for different numbers of antennas per terminal. For a fair comparison, all curves refer to the same overall transmitted power. The constellation is QPSK. The overall number of transmit/receive antennas is also the same in all cases. More specifically, we used the following combinations: \( n = 100 \) and \( n_T = 1 \) (dotted line), \( n = 50 \) and \( n_T = 2 \) (dashed line), and \( n = 25 \) and \( n_T = 4 \) (solid line). The connectivity is insured with probability \( p_{\text{con}} = 0.99 \) and the out-of-service event refers to a target BER of \( 10^{-3} \) and it is required to occur with a maximum time percentage of \( P_{\text{out}} = 10^{-2} \).

We can see, from Figure 2, the advantage of diversity that makes in this case \( D_X(x) \) is strictly monotone.

The connectivity is also a function of the bit rate. As an example, in Figure 3 we show the minimum SNR required for connectivity, assuming an efficiency of 2, 4, and 6 bits/sec/Hz, achieved using 4, 16, or 64-QAM constellations. As expected, an increase of rate requires an increase of node density, for a given power budget. Hence, the bit rate may result as a compromise between the number of antennas, node density, energy per node, and complexity.

3.3. Cooperative Communications

The next question is “What can we do to improve the connectivity if we have only single antenna transceivers?” We can resort to cooperation among nearby terminals. The idea is pictorially sketched in Figure 4, where we see three nodes, A, B, and C. In the absence of any cooperation, each node covers a circle of radius \( r_0 \) given by (13). For example, in Figure 4, A and B are connected, but C is isolated. However, if nodes A and B cooperate, they can give possible a considerable decrease of the nodes density. Clearly, terminals with multiple antennas provide a larger coverage than single antenna terminals, but at the cost of increased complexity.

Fig. 4. Coverage in cooperating networks.
4 (b), where the bigger circle is the area covered by a system located in the center of gravity of the nodes $A$ and $B$, with a bigger radius resulting from the use of a MISO system with two transmit and one receive antenna. Proceeding as in the previous section, denoting with $n_{\text{relay}}$ the number of cooperating (relay) nodes, we have the potential of diversity gain $n_{\text{relay}} + 1$ (the relays plus the source itself). The existence of the bigger circle is a result of the cooperation between $A$ and $B$. Thanks to cooperation, a disconnected network may become connected, as shown in the example of Figure 4 (b), using the same overall transmit power.

Let us now quantify how much is the coverage increase, due to cooperation, and how this affects the connectivity. If we have $n_{\text{relay}}$ relays cooperating with a source and a destination with one receive antenna, the (maximum) diversity gain is $n_T := n_{\text{relay}} + 1$. Repeating derivations similar to the ones described in the previous section, with the only exception that now we only have transmit diversity, but no receive diversity because each receiver has a single antenna, the coverage radius in case of cooperation is

$$r_{\text{coop}} \simeq \left( \frac{g_M E_b}{N_0 T \log(C_M/F_b)} \right)^{1/m} = \beta r_{\text{cov}},$$  \hspace{1cm} (25)

where $r_{\text{cov}}$ is given by (14) and

$$\beta := \left( \frac{(n_T!)^{1/n_T}}{n_T P_{\text{out}}^{(n_T-1)/n_T}} \right)^{1/\lambda}.$$  \hspace{1cm} (26)

Therefore, the coverage increases by a factor $\beta$ that depends on the number of cooperating nodes and on the desired out-of-service probability.

The effect of the coverage increase on the network connectivity is illustrated in Figure 5, where we report the connection probabilities obtained without cooperation (dashed line) and with cooperation (solid line). In case of cooperation, we considered only the case of no more than two cooperating terminals. The probabilities shown in Figure 5 have been estimated over a set of 200 independent network realizations. As a comparison term, we report in Figure 5 the connection probability of a non-cooperative network, but having a coverage radius $\beta r_{\text{cov}}$ (dotted line). We can see that cooperation between pairs of radio nodes is sufficient to yield an SNR gain of approximately seven dB. Interestingly, the curve obtained without cooperation, but with a coverage radius $\beta r_{\text{cov}}$ has approximately the same connectivity as the cooperative case, where the coverage of each node is $r_{\text{cov}}$. Recalling, from (25), that the transmitted power is proportional to $r^2$, if $r$ is the coverage, we infer that cooperation among pairs of terminals yields an improvement in terms of transmitted power approximately equal to $\beta^2 r^2$, when $n_T = 2$.

4. CONCLUSION

In conclusion, the approximate expressions derived in this paper have been shown to fit quite well the behavior of wireless networks having a random topology, modeled as a 2D Poisson point process, in terms of connectivity. The formulas, albeit approximate, help to predict the gain achievable by using multiantenna terminals, either real or virtual, i.e., through cooperative communications. We have derived expressions for the density with which the network nodes should be deployed in order to guarantee the network connectivity with a desired probability. In this paper, we have only shown results concerning Rayleigh fading channels. We have extended the analysis to Nakagami-$m$-fading channels and we showed that the advantage decreases as the index $m$ increases, i.e., as the channel tends to be less and less random.

5. REFERENCES