Channel models and modes

Sergio Barbarossa

University of Rome “La Sapienza”
Information and Communication Department
Via Eudossiana 18, 00184 Rome, Italy

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Chapter 1

Channel models and modes

1.1 INTRODUCTION

Appropriate modeling of multiple-input/multiple-output (MIMO) channels is naturally the first step in the study of multiantenna systems. Before starting our analysis, it is important to clarify that, even though MIMO channel systems are usually intended to be multiple antenna systems, this is not always the case. Multiple antennas are necessary only to create multiple spatial channels, but we should keep in mind that other dimensions can exploited as well, such as time, frequency and polarization. For example, multiple temporal channels can be implemented sending our information bits through different time slots, as in Time Division Multiple Access (TDMA) systems; frequency channels can be induced by sending bits through different (sub-)carriers, as in Frequency Division Multiple Access (FDMA) systems. Finally, since in wireless communications the transmitted signals are carried by electromagnetic fields, which are vectors characterized by a certain polarization, we can also associate different bits to different polarization states, so as to create multiple polarization channels. Time and frequency variables are not independent as they are related through a Fourier Transform (FT), so that temporal and frequency channels are different ways of managing the time dimension. More generally, one can also associate different bits to different codes, as it happens in Code Division Multiple Access (CDMA) systems, so that TDMA, FDMA and CDMA can all be seen as different ways to handle the time domain (or associated domains, like frequency or code domain). Hence, in general, we have three independent domains we can play with to create multiple channels: space, time (or frequency or code) and polarization. Even more interestingly, we can combine the channels induced in different domains so as to create multiple channels in a joint space-time-polarization domain. This is indeed a possibility not far from practical implementations, as it is already under consideration for the incoming third generation (3G) systems, as we will show in this book.
In general, the choice of the domains that we are going to exploit depends on several aspects, including complexity, portability, cost, and so on. For example, incorporating multiple antennas on a portable cellular phone poses some implementation problems, because of the phone dimensions. In fact, most of the potential advantages “promised” by multiple antenna systems are really achieved only if the channels are statistically independent. This requires that the antenna elements are spatially separated by more than half a wavelength. Working, for example, with a carrier frequencies of 1 GHz, the wavelength is 30 cm, so that placing elements more distant than half a wavelength on an handset phone can be troublesome. In such a case, one could place the elements at less than half a wavelength, thus sacrificing part of the benefits, or use polarization instead of space. The price to be paid in this case is the need of using a receive antenna able to catch all polarization components of the electric field, bit the system can achieve a significant improvement without requiring an increase of the phone size. The idea of exploiting the polarization component to induce polarization diversity was proposed for example in [1].

The other dimension one could use, besides space and polarization, is time. The wireless channel is intrinsically time-varying and this property is usually seen as an annoying aspect. However, time variability can be turned into a usual source of diversity, if the transmitter is properly designed.

Clearly, the first step in the design of a system capable of exploiting space, time and polarization consists in a proper modelling of the channel. We start with a Single-Input/Single-Output (SISO) channel, to introduce the basic functions characterizing the channels in the time/frequency plane. We will consider both deterministic as well as statistical channel models. Then we will extend our modelling to the MIMO case.

After having illustrated the main channel models for both SISO and MIMO structures, we will introduce the concept of channel mode and show that the modes are the fundamental vehicles to increase transmission rate and/or diversity.

1.2 WAVEFIELDS

In this chapter we review a few basic concepts about electromagnetic fields and propagation. The reader interested in a more in-depth treatment of this subject is invited to check, for example [2] or, for a direct link to multiantenna systems, [3].

The electromagnetic field is a vector field and it is described, in each point of space, by two vectors, the electric and magnetic fields. Each field is described by a vector which is a function of both space and time. Indicating with \( \text{vec}x_0, \text{vec}y_0, \) and \( \text{vec}z_0 \), the unit vectors of a cartesian reference system, the electric field in a point
of position \( \vec{r} := \vec{r}_0 x + \vec{r}_0 y + \vec{r}_0 z \) can be written as

\[
\vec{e}(\vec{r}; t) = \vec{e}_x(\vec{r}; t) + \vec{e}_y(\vec{r}; t) + \vec{e}_z(\vec{r}; t).
\] (1.1)

If we consider a monochromatic wave, characterized by the wavelength \( \lambda \), with propagation vector \( \vec{v} \), the electric field varies as a function of space and time as follows

\[
\vec{e}(\vec{r}; t) = \vec{E}_0 e^{j(\omega t - \vec{v} \cdot \vec{r})},
\] (1.2)

where \( \vec{E}_0 \) is the electric field in \( \vec{r} = 0 \), at \( t = 0 \). The propagation vector, in such a case, is \( \vec{v} = 2\pi/\lambda \vec{n} \), where \( \vec{n} \) is the unit vector indicating the direction of propagation.

In particular, if we consider wave propagation in a medium with no charges nor currents, solving Maxwell’s equation it turns out that the electric field vector \( \vec{e}(\vec{r}; t) \) and the associated magnetic field \( \vec{h}(\vec{r}; t) \) are orthogonal to each other and they are also both orthogonal to the direction of propagation, established by the vector \( \vec{v} \). More specifically,

\[
\vec{v} \cdot \vec{e}(\vec{r}; t) = 0,
\] (1.3)

where \( \cdot \) denotes scalar product and

\[
\vec{h}(\vec{r}; t) = \frac{1}{Z_0} \vec{n} \times \vec{e}(\vec{r}; t),
\] (1.4)

where \( \times \) denotes vector product, whereas \( Z_0 \) is the intrinsic impedance of free-space, equal, approximately, to 377 ohms.

We express this property saying that the propagating wave is a transverse electromagnetic (TEM) field. In a TEM field, the electric and magnetic fields lie on a plane perpendicular to the direction of propagation. In particular, let us assume to have a radiating antenna of size \( a \). We choose the center of our reference system as the center of this radiation source. The field in the immediate neighborhood of the source is composed of several different components. Each component is characterized by a specific attenuation as the component travels away from the source. At a certain distance from the antenna, in a region defined as the antenna far-field many components can be neglected and the only significant field components attenuate as \( 1/r \), where \( r \) is the distance from the antenna. The antenna far-field is conventionally defined as the region of points distant more than

\[
R_F = \frac{2a^2}{\lambda}
\] (1.5)

from the antenna. Introducing a polar reference system centered on the radiating
source and denoting with $E_r(r; t)$, $E_\theta(r; t)$, and $E_\phi(r; t)$ the three components of the electric field in the far-field region, from (1.3) we have $E_r = 0$. Furthermore, we can write, in general (see, e.g. [2])

$$E_\theta = E_0 e^{-jk_0 r}, \quad E_\phi = \alpha e^{i\beta} E_0 e^{-jk_0 r},$$

(1.6)

where $\alpha$ and $\beta$ are real. Assuming, without any loss of generality, that $E_0$ is also real, we can also write the relationship between $E_\theta$ and $E_\phi$ as follows [2]:

$$\left(\frac{E_\phi}{\alpha}\right)^2 + E_\theta^2 = \frac{2\cos\beta}{\alpha} E_0 E_\phi = \frac{E_0^2 \sin^2 \beta}{(4\pi r)^2}.$$  

(1.7)

This is the equation of an ellipse. At any given point in space, the electric field traces out an ellipse, as time passes. A pictorial description of the electromagnetic field is sketched in Figure 1.1, where we report the electric and magnetic fields in different point of space. In each point, the vector rotates as a function of time.

The meaning of polarization can be explained thinking of the action of the electric field on a charged particle. If there is an electrically charged particle at $r$, with charge $q$, immersed in an electric field $\vec{e}(r'; t)$, the particle is subject to a force $\vec{f}(r'; t) = q \vec{e}(r'; t)$. Therefore, the particle tends to oscillate in a direction specified by the direction of the electric field vector, with an intensity given by the product of the electric field intensity (the modulus of $\vec{e}(r'; t)$) times the particle charge. Equation (1.7) can degenerate into a circle or a straight line, in
which cases we say that we have circular or it linear polarization. In particular, we have a circularly polarized wave when \( \alpha = 1 \) and \( \beta = \pm \pi/2 \). Conversely, we have a linear polarization when \( \beta = k\pi \), with \( k \) integer, which implies \( E_\beta = \alpha E_\phi \).

As already mentioned in the introduction of this chapter, polarization can be very important because it may be exploited to obtain multi-mode propagation even using a single antenna at both transmitter and receiver, as suggested in [1]. We can in fact associate different information bits with the two polarization components of the transmitted electric field. Then, at the receiver, we need to be able to measure at least two polarization components. In general, unfortunately, because of the complicated interaction of the transmitted field with the propagation environment, the electric field impinging on the receive antenna is not necessarily on the same plane as the transmitted field. We introduce here a completely general relationship. However, just because of its generality, this formulation does not give much insight into the problem. For this reason, in the next sections we will consider more specific cases, which help to better understand the physical properties underlying the channel input/output relationship.

Let us indicate with \( \mathbf{vece}_t(r' ; t) \) the electric field transmitted from an antenna placed at \( r' \) and with \( \mathbf{vece}_r(r; t) \) the electric field impinging on the receive antenna, located at \( r \). Assuming linear propagation, the relationship between the transmitted and received fields is

\[
\mathbf{vece}_r(r; t) = \int_S \int_{t' = -\infty}^t K(r, r' ; t, t') \mathbf{vece}_t(r' ; t') dr' dt',
\]

where the first (triple or volume) integral is made over space, and more specifically over the domain \( S \) indicating the region comprising all sources of radiation, whereas the second integral is made over time. The integration limits from \(-\infty\) to \( t \) reflects the causality property, as the field \( \mathbf{vece}_r(r; t) \) at any instant \( t \) depends only on the past, i.e. on fields radiating up to the instant \( t \) itself, but not on the future.

The matrix \( K(r, r' ; t, t') \) is a matrix kernel with components

\[
K(r, r' ; t, t') = \begin{pmatrix}
K_{xx}(r, r' ; t, t') & K_{xy}(r, r' ; t, t') & K_{xz}(r, r' ; t, t') \\
K_{yx}(r, r' ; t, t') & K_{yy}(r, r' ; t, t') & K_{yz}(r, r' ; t, t') \\
K_{zx}(r, r' ; t, t') & K_{zy}(r, r' ; t, t') & K_{zz}(r, r' ; t, t')
\end{pmatrix},
\]

If \( K(r, r' ; t, t') \) is diagonal, the received field is parallel to the transmitted field, otherwise there is a cross-polarization effect. In a dense scattering environment, for example, we observe cross-polarization due to the superposition of several reflections, scattering and diffraction effects. The interactions of the waves with the environment are so complicated that it is not meaningful to try to write a deterministic relationship between the transmitted and received fields. For this reason, we will proceed, in Section 1.3.2, to provide a statistical characterization of the channel.
1.3 SISO CHANNELS

Let us start considering a SISO channel. This constitutes a particular case of (1.8), occurring when the source of radiation and the receiving antenna are both pointlike and we consider only one polarization. In such a case, we may concentrate on the time coordinates, because there is no integration over the space dimension. Denoting with $x(t)$ the (scalar) signal sent through the channel and with $y(t)$ the corresponding received signal, any linear channel can be described by the following input/output relationship:

$$ y(t) = \int_{-\infty}^{\infty} h(t, \tau)x(t - \tau)d\tau \quad (1.10) $$

where $h(t, \tau)$ is the time-varying channel impulse response. With respect to the more general relationship (1.8), (1.10) does not have an explicit dependence over the space dimension. In this case, in fact, since both source and destination are pointlike and we are using only one polarization, there is no integration over space. The impulse function $h(t, \tau)$ depends on the coordinates of the transmitting and receiving antennas, but since there is no integration over space, to keep the notation as simple as possible, we neglect the dependence of $h(t, \tau)$ from the spatial coordinates.

In general, given the high complexity of the physical interactions characterizing the transmission through a real channel, including reflections, scattering, refraction or diffractions (see, e.g. [2]), the most appropriate modelling of the impulse response is probabilistic, so that $h(t, \tau)$ is a 2D random process. However, in several cases of practical interest, the realizations of this random process can be described very accurately by a parametric model whose parameters represent physically meaningful quantities, such as delays, Doppler frequencies and reflection coefficients. Therefore, both deterministic and stochastic approaches are equally useful in describing a time-varying channel, even though they embrace different aspects: The stochastic model is better suited for describing global behavior, whereas the deterministic one is more useful to study the transmission through a specific channel realization. Modelling the parameters of a multipath channel as random variables, for example, provides a simple yet important random channel model. Parametric modelling is especially important in devising channel estimation and tracking algorithms (see e.g. [4]). For all these reasons, in the next section we start with a deterministic characterization; the random model counterpart will be studied in the ensuing section.
1.3.1 Deterministic models

1.3.1.1 Continuous-time model

A SISO channel can be described equivalently by one out of four interchangeable functions. Following the same notation introduced in the pioneering work of Bello [5], any linear SISO channel can be fully described by its impulse response \( h(t, \tau) \), which was introduced in (1.10), or by any of the following characteristic system functions:

i) time-varying transfer function

\[
H(t, f) := \int_{-\infty}^{\infty} h(t, \tau) e^{-j2\pi f \tau} d\tau;
\]  

(1.11)

ii) delay-Doppler spread function

\[
S(\nu, \tau) := \int_{-\infty}^{\infty} h(t, \tau) e^{-j2\pi \nu \tau} dt;
\]  

(1.12)

iii) output Doppler-spread function

\[
Q(\nu, f) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t, \tau) e^{-j2\pi (\nu t + f \tau)} dt d\tau.
\]  

(1.13)

The relationship between these functions is depicted in Figure 1.2. In the figure, the symbol \( F_{1D} \) indicates one-dimensional Fourier Transform (FT), whereas \( F_{2D} \) stands for two-dimensional Fourier Transform (the variables within brackets indicate the pair of dual variables related by a FT, as established by (1.12, 1.11), and 1.13).

Example: Multipath channel

Let us consider the multipath model depicted in Figure 1.3, where the receiver gets a superposition of replicas of the transmitted signal. We consider the situation where the transmitter is fixed and the receiver is moving with constant speed \( v \) along the direction specified by the angle \( \gamma \). We assume, for simplicity, that the transmitted signal is mono-chromatic, with carrier frequency \( f_0 \). Each path in Figure 1.3 is characterized by a triplet of values: the amplitude \( h_k \), the distance \( r_k \) and the angle \( \theta_k \). We further assume that all these values are constant within the observation interval. Since the electromagnetic waves propagate at the speed of light \( c = 3.10^8 \text{m/sec} \), the delay \( \tau_k \) and the Doppler frequency shifts \( f_k \) are related...
Figure 1.2  Relationship between system functions.

Figure 1.3  Multipath channel.
Figure 1.4  Modulus of a multipath channel spread function.

to the distances $r_k$ and the angles $\theta_k$ through the following relationships

$$\tau_k = \frac{r_k}{c}, \quad f_k = \frac{v}{\lambda} \cos(\theta_k - \gamma),$$

where $\lambda = c/f_0$. Hence, denoting with $x(t)$ the complex envelope of the transmitted signal, the corresponding received signal $y(t)$ is

$$y(t) = \sum_{k=0}^{K-1} h_k x(\tau - \tau_k) e^{j2\pi f_k t}. \quad (1.15)$$

Such a system is characterized by the impulse response

$$h(t, \tau) = \sum_{k=0}^{K-1} h_k \delta(\tau - \tau_k) e^{j2\pi f_k t} \quad (1.16)$$
or, exploiting the relationship between the different system functions, by the delay-Doppler spread function

$$S(\nu, \tau) = \sum_{k=0}^{K-1} h_k \delta(\tau - \tau_k) \delta(\nu - f_k), \quad (1.17)$$

the time-varying transfer function

$$H(t, f) = \sum_{k=0}^{K-1} h_k e^{j2\pi(f_k t - \tau_k f)} \quad (1.18)$$
or the Doppler spread function

$$Q(\nu, f) = \sum_{k=0}^{K-1} h_k e^{-j2\pi(\tau_k f)\delta(\nu - f_k)}.$$ \hspace{1cm} (1.19)

As we can see from (1.17), the delay-Doppler spread of multipath channels function assumes then a very specific form, as it is composed only by Dirac pulses. An example of spread function of a multipath channel is reported in Figure 1.4. Delay-

Doppler spread function

Each channel function has its own specific meaning. For example, applying an inverse FT to (1.12, we can express $h(t, \tau)$ in terms of $S(\nu, \tau)$

$$h(t, \tau) := \int_{-\infty}^{\infty} S(\nu, \tau) e^{j2\pi\nu t} d\nu.$$ \hspace{1cm} (1.20)

Then, plugging (1.20) into (1.10), we can write the I/O relationship (1.10) as

$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\nu, \tau) x(t - \tau)e^{j2\pi\nu t} d\nu d\tau.$$ \hspace{1cm} (1.21)

Expressing the integral as the limit value of the series

$$y(t) = \lim_{\Delta\tau, \Delta\nu \to 0} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} S(k\Delta\nu, l\Delta\tau) x(t - k\Delta\tau)e^{j2\pi\nu t},$$ \hspace{1cm} (1.22)

we can state that the output of every linear system can always be written as the superposition of replicas of the input signal shifted in time by $\tau$, in frequency by $\nu$, and scaled in amplitude by the factor $S(\nu, \tau)\Delta\tau\Delta\nu$. The function $S(\nu, \tau)$ assumes thus the meaning of a (complex) amplitude density, quantifying the contribution on the received signal corresponding to Doppler shift $\nu$ and delay $\tau$. In case of a multipath channel, the function $S(\nu, \tau)$ is composed only by Dirac pulses and the output is the sum of a finite number of replicas, whereas in the most general case, $S(\nu, \tau)$ may have finite components plus Dirac pulses.

Since the Doppler shifts and the delay assume finite values, the spread function is necessarily concentrated in a certain region of the $S(\nu, \tau)$ plane. We can quantify the concentration of $S(\nu, \tau)$ by introducing the moments of its modulus. In particular, we define the following absolute moments:

$$m_{S}^{(k,l)} := \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S(\nu, \tau)||\tau - \tau_0|^k|\nu - \nu_0|^l d\nu d\tau}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S(\nu, \tau)| d\nu d\tau},$$ \hspace{1cm} (1.23)
The values $\tau_0$ and $\nu_0$ can be chosen to minimize, for example, $m_S^{(1,0)}$ and $m_S^{(0,1)}$, separately. In such a case, $\tau_0$ is equal to the median value of $S(\tau) := \int_{-\infty}^{\infty} |S(\nu, \tau)| d\nu$, and $\nu_0$ is the median value of $S(\nu) := \int_{-\infty}^{\infty} |S(\nu, \tau)| d\tau$. Alternatively, if $S(\nu, \tau)$ has exactly finite support, we can measure the extent of its support through its area, as

$$ A := \max_{k}(|\nu|) \cdot \max_{k}(|\tau|), \text{ such that } S(\nu, \tau) \neq 0. \quad (1.24) $$

A channel is said to be underspread if $A \ll 1$ or, equivalently, the normalized moments (1.23) are all much smaller than one. This property indicates that the spread function is very concentrated around the origin of the delay-Doppler domain. Interestingly, most communication channels are underspread, as it can be checked through the following example.

Let us consider a narrowband system, transmitting around a frequency $f_0 = 3 \text{GHz}$, with wavelength $\lambda = c/f_0 = 0.1 m$. Let us assume that the maximum delay between the received signal components is $\tau_{\text{max}} = 1 \mu\text{sec}$. This corresponds to a difference in the distance travelled by the fastest and slowest received components equal to $d = c\tau_{\text{max}} = 3 \cdot 10^8 \cdot 10^{-6} = 300 \text{ meters}$. Let us suppose also that the receiver is moving at a speed of $v = 120 \text{ Km/hr}$. The maximum Doppler frequency is thus $\nu_{\text{max}} \approx 666 \text{ Hz}$. Hence, we have $A \approx 6.6610^{-4}$, which is indeed much less than 1.

**Time-varying transfer function**

The time-varying transfer function $H(t, f)$ is particularly useful to describe time-invariant systems. In such a case, in fact, the spectrum of the system output $Y(f)$ is related to the spectrum of the input by the simple relationship

$$ Y(f) = H(f)X(f). \quad (1.25) $$

This equation explains the filtering operation (in the frequency domain) operated by a time-invariant channel. In fact, (1.25) makes immediately evident that the frequency components of the input are more or less attenuated by the transit through the channel depending on the values of the channel transfer function at that frequency. If the channel is time-varying, clearly (1.25) does not hold anymore. However, if we consider a block transmission system, where each block has duration $T$ and we assume that the channel transfer function is constant within each block, we can generalize (1.25) as follows. If we denote by $Y(n; f)$ ($X(n; f)$) the spectrum of the received (transmitted) signal, during the $n$-th time slot, if the channel within each time slot is time-invariant, we can write

$$ Y(n; f) = H(nT, f)X(n; f). \quad (1.26) $$
How fast $H(t, f)$ varies as a function of $t$ and $f$ is dictated by the support of the delay-Doppler spreading function $S(\nu, \tau)$. In fact, $H(t, f)$ is related to $S(\nu, \tau)$ by a two-dimensional Fourier Transform (FT):

$$H(t, f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\nu, \tau)e^{j2\pi(\nu t - f\tau)} d\tau d\nu.$$  \hfill (1.27)

Therefore, $H(t, f)$ is smooth along $t$ ($f$) if the support of $S(\nu, \tau)$ along $\nu$ ($\tau$) is small and vice versa. More specifically, the smoothness of $H(t, f)$ can be quantified, or at least upper bounded, by using the moments of $S(\nu, \tau)$. More specifically, differentiating (1.27) with respect to both $t$ ($k$ times) and $f$ ($l$ times), we have

$$\frac{\partial^{k+l}H(t, f)}{\partial t^k \partial f^l} = (j2\pi)^{k+l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nu^k \tau^l S(\nu, \tau)e^{j2\pi(\nu t - f\tau)} d\tau d\nu.$$  \hfill (1.28)

Therefore,

$$\left| \frac{\partial^{k+l}H(t, f)}{\partial t^k \partial f^l} \right| < (2\pi)^{k+l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nu|^k |\tau|^l |S(\nu, \tau)| d\tau d\nu = (2\pi)^{k+l} m_S^{(k,l)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |S(\nu, \tau)| dv d\tau.$$  \hfill (1.29)

Recalling the definition underspread channel, we may say that the time-varying transfer function is certainly a smooth function. Output Doppler spread function

The function $Q(\nu, f)$ is useful to derive the dual input/output relationship of (1.10) in the frequency domain:

$$Y(f) = \int_{-\infty}^{\infty} Q(f - \nu, \nu)X(\nu) d\nu = \int_{-\infty}^{\infty} Q(\nu, f - \nu)X(f - \nu) d\nu,$$  \hfill (1.30)

where $X(f)$ and $Y(f)$ denote the Fourier transform (FT) of the input and output signals, respectively.

### 1.3.1.2 Discrete-time model

The continuous-time characterization is useful to grasp some channel properties, but, as we are basically interested in digital communications, it is important to derive the discrete-time channel modeling. We consider a system, as depicted in Figure 1.2. We start with a a SISO system and, more specifically, we denote with $x[k]$ the (generally complex) $k$th transmitted symbol and with $g_{T}(t)$ is the transmit
lowpass filter whose bandwidth is directly proportional to the symbol rate $1/T_s$. The
impulse response $g_T(t)$ has a Nyquist characteristic and it is usually a root raised
cosine filter [6]. Using linear modulation, the baseband transmitted signal can be
expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} x[k] g_T(t - kT_s). \tag{1.31}$$

The channel output can then be written as

$$z(t) = \sum_{k=-\infty}^{\infty} x[k] \int_{-\infty}^{\infty} h(t, \tau) g_T(t - \tau - kT_s) d\tau. \tag{1.32}$$

The received signal is demodulated, low-pass filtered and sampled. Denoting by
$g_R(t)$ the impulse response of the receive low-pass filter, the baseband received
signal is

$$y(t) = \sum_{k=-\infty}^{\infty} x[k] \int_{-\infty}^{\infty} g_R(t - \theta) g_T(\theta - \tau - kT_s) h(\theta, \tau) d\theta d\tau. \tag{1.33}$$

Hence, sampling $y(t)$ at symbol period $T_s$, we obtain the sequence

$$y[n] := y(nT_s) = \sum_{k=-\infty}^{\infty} h[n, n - k] x[k], \tag{1.34}$$

where we have introduced the equivalent discrete-time impulse response

$$h[n, n - k] := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_R(nT_s - \theta) g_T(\theta - \tau - kT_s) h(\theta, \tau) d\theta d\tau. \tag{1.35}$$

Equation (1.34) is the discrete-time counterpart of (1.10).

To gain better insight into the transmission through LTV channels, it is useful
to express (1.35) in the frequency domain. Specifically, introducing the transfer
functions $G_T(f)$ and $G_R(f)$ of the transmit and receive filters and using the output
Doppler-spread function (1.13), we may rewrite (1.35) as

$$h[n, n - k] = \int_{-\infty}^{\infty} G_R(f) G_T(\nu) Q(f - \nu) e^{j2\pi (nT_s - \nu)} d\nu df. \tag{1.36}$$

Using now the multipath channel model (1.16), we get

$$h[n, n - k] = \sum_{q=0}^{L} h_q e^{j2\pi f_q T_s} \int_{-\infty}^{\infty} G_R(\nu + f_q) G_T(\nu) e^{j2\pi \nu ((n-k)T_s + \tau_q)} d\nu. \tag{1.37}$$
Substituting the transmit and receive transfer functions $G_T(f)$ and $G_R(f)$ in (1.37), we obtain the equivalent discrete-time impulse response in the most general case. To comprehend some of the basic features of the discrete-time equivalent channels, it is useful to analyze the simple case where both transmit and receive shaping filters are ideal lowpass filters. In particular, setting $G_T(f) = G_R(f) = \sqrt{f_s} rect(f)$, where the rectangular function $rect(f)$ is equal to one for $|f| < 1/2$ and is null otherwise, (1.37) gives rise to the following DT impulse response

$$h[n, k] = \sum_{q=0}^{L} h_q[1 - |f_q T_s| e^{j \pi \nu_q (2n - k + \theta_q)} \sin c[\pi (1 - |\nu_q|)(k - \theta_q)], \quad (1.38)$$

where $[x]^+ \equiv \max(x, 0)$ and we have introduced the normalized delay $\theta_q := \tau / T_s$ and Doppler $\nu_q := f_q T_s$. Since $\max_q |\nu_q| < 1$, it is clear from (1.38) that the components corresponding to higher Doppler shifts $\nu_q$ are more attenuated. This happens because part of their energy falls outside of the receive filter bandwidth. In fact, since the transit through an LTV channel increases the bandwidth of the signal, the receive filter should have a bandwidth greater than $1/T_s$ to keep all the useful energy and, consequently, the sampling rate should also be higher than $1/T_s$, to avoid any loss of information. However, in practice $f_q T_s \ll 1$, so that usually the receive filter bandwidth and sampling rate can be maintained equal to $1/T_s$, without any appreciable loss. As a simple numerical example, using a carrier frequency of 2 GHz in a link of 1 Mbps between terminals in relative motion at a velocity of $v = 150$ Km/hr, the maximum normalized Doppler shift is approximately $2.8 \times 10^{-4}$. This explains why in most practical systems the receiver bandwidth and sampling rates are not higher than $1/T_s$, for the gain obtainable otherwise is not worth of the extra complications related to re-sampling. Therefore, (1.38) can be approximated with negligible error as

$$h[n, k] \approx \sum_{q=0}^{L} h_q e^{j \pi \nu_q (2n - k + \theta_q)} \sin c[\pi (k - \theta_q)]. \quad (1.39)$$

**Ideal channels**

We say that a continuous time (CT) channel is ideal if the channel response $y(t)$ to any input $x(t)$ is a delayed replica of the input, possibly multiplied by a coefficient $A$, so that $y(t) = A x(t - t_0), \forall x(t)$. An ideal channel is then characterized by an impulse response $h(t) = A \delta(t - t_0)$ or, equivalently, by a transfer function $H(f) = A e^{-j 2\pi f t_0}$.

Similarly, we say that a discrete-time (DT) channel is ideal if, for any input sequence $x[n]$, the corresponding output sequence $y[n]$ is a delayed version of $x[n]$. 

possibly scaled in amplitude, i.e. \( y[n] = Ax[n - n_0] \).

**Remark:** An ideal CT channel does not necessarily imply that the corresponding DT channel is ideal. This happens only if the delay \( t_0 \) is an integer multiple of the sampling time \( T_s \).

**Matrix formulation**

If we concentrate on block transmissions, where the information symbols stream is parsed inconsecutive blocks of length, let us say, \( N \), we can collect the transmitted block in a column vector \( \mathbf{x}(n) := (x(nN), \ldots, x(nN + N - 1))^T \). If the DT channel is FIR of order \( L \), the corresponding block of \( N \) received symbols is \( \mathbf{y}(n) := (y(nP), \ldots, y(nP + P - 1))^T \) and the input/output relationship is given by the following matrix equation

\[
\mathbf{y}(n) = \mathbf{H}(n)\mathbf{x}(n),
\]

where \( P = N + L \), \( \mathbf{H}(n) \) is the \( P \times N \) channel matrix, whose structure is

\[
\mathbf{H}(n) = \begin{bmatrix}
    h(n, 0) & 0 & \cdots & \cdots & 0 \\
    h(n + 1, 1) & h(n + 1, 0) & 0 & \cdots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    h(n + L, L) & h(n + L, L - 1) & \cdots & \ddots & \vdots \\
    0 & \vdots & \ddots & \ddots & h(n + P - L, 0) \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & \cdots & 0 & h(n + P, L)
\end{bmatrix}
\]

Typically, in block transmissions, block are separated by a guard interval that facilitates, at the receiver, the elimination of Inter-Block Interference (IBI). Block transmission will be analyzed in detail in Chapter 3.

**1.3.2 Stochastic models**

The deterministic description of a wireless channel introduced in the previous sections is useful to relate the channel parameters to physically meaningful quantities, such as delays, Doppler frequencies and amplitudes of the received signal components. However, such a characterization is specific of a given channel scenario and, as such, it cannot be used to describe a wireless channel, in general. In practice,
the received signal is the result of the interaction of the radiated electromagnetic
wave with the environment between transmitter and receiver. Such an interaction
is too complicated to be described in a deterministic way. Hence, the only way
to describe the general properties of a wireless channel is in probabilistic terms.
More specifically, the impulse response of a wireless channel can be modelled as a
stochastic (random) process, whose parameters (rather than the impulse responses)
are related to the physical quantities. In this more general setup, the deterministic
impulse responses studied in the previous section, can be seen as realizations of this
random process.

We assume that the channel impulse response admits a Fourier Transform
over both variables $t$ and $\tau$, so that we can properly talk, also for random channels,
of the delay-Doppler spread function $S(\nu, \tau)$, the time-varying transfer function
$H(t, f)$ and the output Doppler-spread function $Q(\nu, f)$. Clearly, all these functions
are now random processes.

The first step in the characterization of the channel functions consists in es-
establishing a link between the statistics of the channel functions and the parameters
of the physical channel. In general, this relationship is rather complicated since it
depends on too many parameters. To simplify the problem, it is useful to look at
the problem at different scales, in terms of both space and time. We will start with
first order statistics in the next section and then we will proceed with second order
statistics in the following one.

**First order statistics**

The first order statistics describe the properties of the single random variable ex-
tracted from the random process describing the channel impulse response. Since
these properties depend on several propagation aspects, it is useful (and common
practice) to distinguish between small, medium, and large scale models. Small-scale
propagation model

Small-scale or short term fading is caused by the superposition of several received
signals backscattered from the environment. A typical example is the urban environ-
ment, where the received signal is the superposition of paths induced by reflection,
diffraction, and scattering from walls, cars, and so on. If several paths arrive with
delays and Doppler shifts which differ by less than the system resolution in time and
frequency, we receive, for each pair of delay and Doppler, a superposition of, possi-
ibly, several contributions. Since these contributions arrive from different reflecting
(diffracting) objects, we may well assume that they are statistically independent.
In such a case, if the number of independent contributions is sufficiently high and
there is no contribution clearly dominant with respect to the others, we may invoke
the central limit theorem and state that each path is characterized by a complex
amplitude modelled as a complex Gaussian random variable \( x_r + jx_i \). This random variable (rv) may have a deterministic component, when there is, for example a line-of-sight (LOS) path. In such a case, the rv has a nonnull mean, otherwise the mean is null. In the latter case, the received complex variable has an amplitude \( r = \sqrt{x_r^2 + x_i^2} \) which has a Rayleigh probability density function (pdf)

\[
p_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} u(r)
\]

(1.42)

where \( 2\sigma^2 \) is the variance of the complex Gaussian rv and \( u(r) \) is the unit step function. This model is known as the Rayleigh fading model.

If there is a LOS path, with amplitude \( A \), the amplitude \( r \) has a Rice distribution, i.e.

\[
p_R(r) = \frac{r}{\sigma^2} e^{-(r^2+A^2)/2\sigma^2} I_0 \left( \frac{Ar}{2\sigma^2} \right) u(r)
\]

(1.43)

where \( I_0(x) \) is the modified Bessel function of the first kind and zero-th order. To analyze the behavior of the Rice pdf, it is useful to introduce the so called Rice factor, defined as

\[
K := \frac{A^2}{2\sigma^2}. \tag{1.44}
\]

In Figure 1.5 we report a few examples of the Rice pdf, for different values of \( K \). As we can see from the figure, for \( K = 0 \), the Rice pdf is equal to the Rayleigh pdf, whereas, as \( K \) increases, the Rice pdf tends to a Gaussian pdf. One more pdf which is sufficiently versatile to include several situations is the Nakagami pdf

\[
p_R(r) = \frac{2m}{\Gamma(m)} \frac{m^{2m-1}}{\sigma^{2m}} e^{-mr^2/(2\sigma^2)}, \quad m \geq \frac{1}{2},
\]

(1.45)

where \( \Gamma(x) \) is the Gamma function, with \( \Gamma(m) = (m-1)! \), when \( m \) is an integer number. Some examples of Nakagami pdf’s are reported in Figure 1.6, for different values of \( m \). Changing the value of \( m \), we can model different situations: for \( m = 1 \), we have the Rayleigh pdf, for \( m = 1/2 \), (1.45) reduces to a one-sided Gaussian pdf, whereas for \( m \) going to \( \infty \), (1.45) tends to a Dirac pulse, which indicates no fading. Medium-scale propagation model

The parameters \( m, A \) or \( \sigma \) appearing in the pdf models examined in the previous section are also clearly related to physical properties of the channel as well as on the transmission parameters (wavelength, attenuations, reflections, ...). In general, these parameters are random variables themselves. A typical pdf commonly used to
Figure 1.5 Rice and Rayleigh probability density functions.

Figure 1.6 Nakagami probability density functions.
characterize the medium scale fluctuations is the log-normal pdf, where the received mean power $p$ is itself a random variable, with pdf

$$p_{p_0}(p_0) = \frac{1}{\sqrt{2\pi}\sigma_p} e^{-\frac{(\ln(p_0) - m_p)^2}{2\sigma_p^2}}, \quad (1.46)$$

with $m_p$ and $\sigma_p^2$ are the mean value and variance of $\ln(p_0)$. The log-normal law is commonly used to model fluctuations due to shadowing effects.

**Large-scale propagation model**

The average power received over a large scale model depends primarily on the distance. In general, indicating with $R$ the distance between transmitter and receiver, the average power $p_a$ follows a polynomial law like

$$p_a = c \left( \frac{R}{R_0} \right)^{-\gamma}, \quad (1.47)$$

where $R_0$ is a reference distance, $c$ is independent of $R$ and it depends on the transmit wavelength and on the transmit and receive antenna radiation patterns; $\gamma$ is a path-loss exponent, which is equal to 2 for free space propagation, but it can range from 2 to 6 in indoor or outdoor environments. The relationship between $p_a$ and the parameters of the small and medium scale models is the following: $p_a$ is the average value of $p_0$, as it appears in (1.46).

**Second order statistics**

To describe how the channel parameters fluctuate with time, we need to introduce the second order statistics. We define the mean value of the impulse response

$$m_h(t, \tau) = E\{h(t, \tau)\} \quad (1.48)$$

and its correlation function

$$R_h(t_1, t_2; \tau_1, \tau_2) := E\{h^*(t_1, \tau_1)h(t_2, \tau_2)\}. \quad (1.49)$$

The symbol $E\{\cdot\}$ denotes expected value. We can also introduce the correlation functions of all the other functions describing the random channel. Clearly, because of the relationship between every pair of functions, there is a corresponding relationship between the corresponding correlation functions. For example, the correlation of the delay-Doppler spread function $R_S(\nu_1, \nu_2; \tau_1, \tau_2) := E\{S^*(\nu_1, \tau_1)S(\nu_2, \tau_2)\}$
Channel models and modes

is related to the correlation of the impulse response by the equation

\[ R_h(t_1, t_2; \tau_1, \tau_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_S(\nu_1, \nu_2; \tau_1, \tau_2) e^{j2\pi(\nu_2 t_2 - \nu_1 t_1)} d\nu_1 d\nu_2. \quad (1.50) \]

Similarly, we can introduce the correlation of the time-varying transfer function, which is related to the impulse response correlation by the following transformation

\[ R_H(t_1, t_2; f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_h(t_1, t_2; \tau_1, \tau_2) e^{-j2\pi(f_2 t_2 - f_1 t_1)} d\tau_1 d\tau_2. \quad (1.51) \]

As we can see from the previous expressions, all channel correlation functions are four-dimensional functions and thus it is not simple to represent them and extract meaningful channel parameters. However, in many practical cases, the correlation functions can be described by a two-dimensional function, as we will show next.

Let us consider a channel where the contributions received from different angles are uncorrelated. This assumption is well justified in practice as arrivals from different angles refer to different scattering objects. In such a case, the function \( R_S(\nu_1, \nu_2; \tau_1, \tau_2) \) is equal to zero for \( \nu_2 \neq \nu_1 \). We can thus describe this correlation function as

\[ R_S(\nu_1, \nu_2; \tau_1, \tau_2) = R_S(\nu_1; \tau_1, \tau_2) \delta(\nu_2 - \nu_1). \quad (1.52) \]

Inserting such an expression in (1.50), we get

\[
R_h(t_1, t_2; \tau_1, \tau_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_S(\nu_1; \tau_1, \tau_2) \delta(\nu_2 - \nu_1) e^{j2\pi(\nu_2 t_2 - \nu_1 t_1)} d\nu_1 d\nu_2 \\
= \int_{-\infty}^{\infty} R_S(\nu_1; \tau_1, \tau_2) \delta(\nu_2 - \nu_1) e^{j2\pi(\nu_1(t_2 - t_1))} d\nu_1. \quad (1.53)
\]

The last term reveals that \( R_h(t_1, t_2; \tau_1, \tau_2) \) depends on \( t_1 \) and \( t_2 \) only through their difference \( \Delta t \), so that we can write

\[ R_h(t_1, t_2; \tau_1, \tau_2) = R_h(\Delta t; \tau_1, \tau_2). \quad (1.54) \]

A channel satisfying (1.54) is a wide-sense stationary (WSS) channel. Therefore, a WSS channel is simply a channel where the contributions referring to different angles (Doppler frequencies) are uncorrelated.

Proceeding in a similar manner, if the contributions from different distances (e.g., delays) are uncorrelated, i.e.

\[ R_S(\nu_1, \nu_2; \tau_1, \tau_2) = R_S(\nu_1, \nu_2; \tau_1) \delta(\tau_2 - \tau_1), \quad (1.55) \]
the correlation of the impulse response assumes the form

$$R_h(t_1, t_2; \tau_1, \tau_2) = R_h(t_1, t_2; \tau_1)\delta(\tau_2 - \tau_1).$$  \hspace{1cm} (1.56)

A channel satisfying (1.56) is denoted as uncorrelated scattering (US) channel. This terminology is largely widespread and this is the only reason we use it here. Nevertheless, it is worth pointing out that in this case, uncorrelated scattering refers only to different delays (and not necessarily Doppler shifts).

Combining the WSS and US properties, a channel where contributions with different delays and Doppler frequencies are uncorrelated is a WSS-US channel and its impulse response is characterized by a correlation function

$$R_h(t_1, t_2; \tau_1, \tau_2) = R_h(\Delta t; \tau_1)\delta(\tau_2 - \tau_1),$$  \hspace{1cm} (1.57)

where $\Delta t = t_2 - t_1$. Therefore, we can state that the correlation properties of a WSS-US channel are fully described by a two-dimensional, rather than four-dimensional, function, which we denote now as $R_h(\Delta t; \tau)$, where the meaning of the two independent variables $t$ and $\tau$ is as specified in the previous derivations. To single out a few parameters, it is useful to analyze the behavior of $R_h(\Delta t; \tau)$ along the axes $\Delta t = 0$ and $\tau = 0$. More specifically, the function $P_h(\tau) := R_h(0, \tau)$ is the power delay profile, as it describes the power behavior as a function of the delay (considering contributions at the same instant, i.e. $t_2 = t_1$ or $\Delta t = 0$). We assume that there are no contributions with negative delays, so that $P_h(\tau) = 0$ for $\tau < 0$.

Two parameters are useful to grasp some of the main features of $P_h(\tau) = 0$, namely the average delay

$$\bar{\tau} := \frac{\int_{-\infty}^{\infty} \tau P_h(\tau) d\tau}{\int_{-\infty}^{\infty} P_h(\tau) d\tau}$$  \hspace{1cm} (1.58)

and the delay spread

$$\sigma_\tau := \left[ \int_{-\infty}^{\infty} (\tau - \bar{\tau})^2 P_h(\tau) d\tau \right]^{1/2}.$$  \hspace{1cm} (1.59)

Example

A typical behavior of the power delay is exponential, i.e.

$$P_h(\tau) = P_0 e^{-\tau/\tau_0} u(\tau),$$  \hspace{1cm} (1.60)

where $u(\tau)$ is the unit step function. In such a case, the average delay is $\bar{\tau} = \tau_0$ and the delay spread is also $\sigma_\tau = \tau_0$. 
So far, we have only considered the correlation properties of the channel impulse response. But, clearly, if the correlation of the channel impulse response has a certain structure, this induces a corresponding structure on the correlation of the other system functions. For example, if the channel is WSS-US, and thus its impulse response correlation is as in (1.57), the correlation of the delay-Doppler spread function is
\[
R_S(\nu_1, \nu_2; \tau_1, \tau_2) = R_S(\nu_1, \tau_1)\delta(\tau_2 - \tau_1)\delta(\nu_2 - \nu_1),
\] (1.61)
and the correlation of the transfer function is
\[
R_H(t_1, t_2; f_1, f_2) = R_H(t_2 - t_1, f_2 - f_1).
\] (1.62)

The function \(R_S(\nu_1, \tau_1)\), for a WSS-US channel, has a direct physical meaning, as it represents the power spectral density gain of the channel at Doppler frequency \(\nu_1\) and delay \(\tau_1\). In fact, from (1.21), we can state that the contribution to the received signal \(y(t)\), coming from the transmitted signal \(x(t)\), within the delay-Doppler interval \([\nu - \Delta\nu/2, \nu + \Delta\nu/2; [\tau - \Delta\tau/2, \tau + \Delta\tau/2]\) is
\[
y_{\nu,\tau}(t) \approx S(\nu, \tau)x(t - \tau)e^{j2\pi\nu t} \Delta\nu\Delta\tau.
\] (1.63)
Thus, the power spectral density of \(y_{\nu,\tau}(t)\), as a function of both delay and Doppler shift is \(E\{|S(\nu, \tau)|^2\}P_x = R_S(\nu, \tau)\).

Exploiting the relationship between spread function and time-varying transfer function, we can also evaluate \(R_H(t_2 - t_1, f_2 - f_1)\) as
\[
R_H(\Delta t, \Delta f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_S(\nu, \tau)e^{j2\pi(\Delta\nu t - \Delta f \tau)} d\nu d\tau,
\] (1.64)
where \(\Delta t := t_2 - t_1\) and \(\Delta f := f_2 - f_1\).

**Physical channel models**

So far, we have described the main channel correlation properties, but without providing any explicit behavior. In this section, we recall some physically meaningful models. We start with narrowband channels and then we will extend the analysis to wideband channels.

**Narrowband models**

We start with a flat fading channel. In this case, all the contributions arrive with the same delay. More precisely, all contributions arrive with delays that differ from each
by less than the capability of the systems to discriminate them in time. The channel impulse response assumes, in such a case, the form

$$h(t, \tau) = h(t) \delta(\tau - \tau_0),$$

(1.65)

where \( \tau_0 \) represents the delay. In this case, we can concentrate on \( h(t) \). We recall now the so called Jakes’ model, which is commonly employed to characterize flat fading narrowband channels. The basic assumptions underlying such a model are that: i) the receiver has an omni-directional antenna, so that the power spectral density \( P_\theta(\theta) \) arriving at the receiver as a function of the angle \( \theta \) is distributed uniformly; ii) the scatterers present in the communication channel are fixed so that they do not introduce further Doppler effects. Assumption i) simply means that, since in general there is no reason to expect more power from a certain angle with respect to other angles, it is perfectly justifiable to assume that \( P_\theta(\theta) \) does not vary with \( \theta \). With reference to Figure 1.3, where \( \nu \) denotes the velocity vector of the receiver and \( \gamma \) is the angle between the direction of the velocity vector and the line-of-sight (LOS) between transmitter and receiver, the relationship between the Doppler shift \( \nu \) and the angle \( \theta \) is, as in (1.14),

$$\nu(\theta) = \frac{\nu}{\lambda} \cos(\theta - \gamma).$$

(1.66)

This relationship establishes a corresponding relationship between the received power spectral distribution as a function of \( \theta \) and the distribution as a function of \( \nu \). Specifically, we have

$$P_\nu(\nu) = \sum_i P_\theta(\theta_i(\nu)) \left| \frac{\partial \theta_i(\nu)}{\partial \nu} \right|_{\theta = \theta_i(\nu)},$$

(1.67)

where \( \theta_i(\nu) = \gamma \) \( \pm \arccos(\lambda \nu / \nu) \) denotes the i-th inverse solution of (1.66). In the interval \([-\pi, \pi]\], clearly there are two inverse solutions. Thus, the power spectral density, as a function of \( \nu \), is

$$P_\nu(\nu) = \frac{P_0}{\pi} \frac{1}{\sqrt{f_{\text{max}}^2 - \nu^2}}, \quad \nu \in [-f_{\text{max}}, f_{\text{max}}],$$

(1.68)

where \( f_{\text{max}} := \nu / \lambda \). Starting from \( P_\nu(\nu) \), we can compute the correlation of the transfer function as the inverse Fourier Transform of \( P_\nu(\nu) \) and the result is known in closed form

$$R_H(\Delta t, 0) = P_0 J_0(2\pi f_{\text{max}} \Delta t).$$

(1.69)

The corresponding correlation coefficient, i.e. \( \rho_H(\Delta t) := R_H(\Delta t) / R_H(0) \) is depicted in Figure 1.7. From this figure, we notice that the values of \( H(t, f) \) are uncorrelated when the separation in time \( \Delta t \approx 0.4 / f_{\text{max}} \).
Wideband models

Wideband systems are capable of discriminating multipath arrivals both in time and in frequency. In particular, denoting with $B$ the system bandwidth and with $T$ the duration of the observation, the system is capable of discriminating arrivals whose delays differ for more than $\delta \tau = 1/B$ and/or whose Doppler frequency differ more than $\delta f = 1/T$. This discrimination capability implies that all the space around transmitter and receiver can be tasselled according to a grid such that each tassel contains points which are not distinguishable, at the receiver, neither in terms of delay nor in terms of Doppler frequency. The geometry is sketched in Figure 1.8. All the points located over the ellipses having as foci the transmitter and receiver locations give rise to the same delay, whereas all the points located over a straight line departing from the receiver give rise to the same Doppler frequency. With infinite resolution capability (i.e. for $B = \infty$ and $T = \infty$), each point would have its own delay and Doppler coordinates. However, in the practical case where both time and frequency resolution are nonnull, there is a region of finite nonnull area (tassel) which give rise to returns which are not distinguishable at the receiver. With reference to Figure 1.8, all points located within the dashed tassels around the receiver are indistinguishable in terms of delay or Doppler. The tessellation
sketched in Figure 1.8 refers to the scenario where the following assumptions hold true: i) the only moving terminal is the receiver; ii) there are no moving objects between transmitter (Tx) and receiver (Rx); iii) each path arrives with a single bounce (multiple-bounce arrivals are neglected).

The geometry depicted in Figure 1.8 established a relationship between the scatterers’ distribution and the received power density, as a function of delay and Doppler. That geometry was first proposed in [7], [8] to find out a relationship between the received power and the Doppler frequency or angle of arrival. This same geometry can be used to derive a closed form expression for the received power density. In particular, assuming that i) the scatterers are uniformly distributed around the receiver, within a circle of radius \( R \), and ii) the backscattering coefficient is constant within the circle of radius \( R \), we can say that the power contribution coming from each tassel is proportional to the tassel area, with a proportionality coefficient equal for all the tassels. As a consequence of these assumptions, the received power as a function of delay \( \tau \) and Doppler shift \( \nu \) is:

\[
\Delta P(\tau_k, \nu_k) = \frac{c}{2} \frac{\varepsilon^2 \tau_k^2}{\tau_k^2} + c \tau_k d \cos \theta_k + \frac{c^2}{(c^2 + d \cos \theta_k)^2} \Delta \nu_k \Delta \tau_k
\]

(1.70)

with \( \theta_k = \gamma \pm \cos^{-1}(f_k/f_M) \).

As an example, in Figure 1.9 we report the received power, as a function of the normalized Doppler frequency \( \nu/f_{max} \), where \( f_{max} = v/\lambda \), for two different values of \( \tau \): \( \tau_0 = 0 \) and \( \tau_1 = T \). We set in this case \( \gamma = \pi/2 \) and we assumed a radius \( R \) of 150 meters. We compare the theoretical behavior (solid line) obtained with (1.70) and the results of a simulation program (dashed line) that generates
Figure 1.9  Power density as a function of $\nu$, for two different values of $\tau$: a) $\tau = 0$, b) $\tau = T_s$; $\gamma = \pi/2$; theoretical values (solid line) and simulations results (dashed line).
the presence of a certain number of scatterers, distributed uniformly within the circle around the receiver and computes the received power. Interestingly, we notice that when \( \tau = 0 \), we have contributions from all the Doppler shifts, whereas for \( \tau > 0 \), there are no returns for very small Doppler frequencies. The reason for this behavior is evident from Figure 1.8, where we can observe that, for low greater than a threshold, there are directions (around the line of sight line) where there are no points inside the circle. Therefore, for Doppler shifts corresponding to those directions there is no received power. Clearly, the power density depends on the direction of the receiver velocity vectors. As an example, in Figure 1.9, we report the result obtained in the same conditions as in Figure 1.8, except for \( \gamma = \pi/4 \). Clearly, if we integrate the received power over all values of \( \tau \), we should receive a power density vs, frequency coinciding with the behavior predicted by Jakes’ model. We can check this property in Figure 1.11, where we report the summation, for every \( \nu \) of the powers received from all values of \( \tau \), for both the situations of \( \gamma = \pi/2 \) and \( \gamma = \pi/4 \). From Figure 1.11 we see that indeed the power behavior follows the law given by Eqn. (1.68). The previous model has a direct physical interpretation, but it is also based on a series of assumptions, such as the stationarity of the scatterers surrounding the receiver antenna, which could not be met in a practical case. Another model, easier to deal with from the mathematical point of view, assumes that the power density, as a function of delay and Doppler, has the following behavior

\[
R_s(\nu, \tau) = \frac{P_0}{2\pi \nu_0 \sqrt{f_{max}^2 - \nu^2}} e^{-|\tau|/\tau_0},
\]

(1.71)

where \( P_0 \) is the overall received power. The two-dimensional Fourier Transform of this function is known in closed form, so that we can say that in this case, the correlation of the time-varying transfer function is

\[
R_H(\Delta t, \Delta f) = P_0 \frac{J_0(2\pi f_{max} \Delta t)}{1 + (2\pi \Delta f \tau_0)^2}.
\]

(1.72)

### 1.3.3 Channel classification

The correlation of the time-varying transfer function is an important function, as it allows to introduce two important parameters: the channel coherence time and bandwidth. Roughly speaking, we define the channel coherence time as the duration of the time interval which which the channel may be assumed to be approximately constant. In formulas, the channel coherence time \( T_c \) is the value such that \( R_H(T_c, 0) = R_H(0, 0)/2 \). For example, assuming (1.72) to be valid, we have

\[
T_c \approx \frac{1}{2f_{max}}.
\]

(1.73)
Figure 1.10  Power density as a function of $\nu$, for two different values of $\tau$: a) $\tau = 0$, b) $\tau = T_\xi$; $\gamma = \pi/4$; theoretical values (solid line) and simulations results (dashed line).
Therefore, the channel coherence time is roughly inversely proportional to the maximum Doppler frequency.

Similarly, we define the channel coherence bandwidth as the value $B_c$ such that $R_H(0, B_c) = R_H(0, 0)/2$. Again, assuming (1.72), the coherence bandwidth is

$$B_c = \frac{1}{2\pi\nu_0}. \quad (1.74)$$

As we will see in the next sections, the concepts of channel coherence time and bandwidth play a fundamental role in the design of proper transmission schemes. Depending on their value and, more specifically, on the relationship between these values and the time and bandwidth of the signals transiting through the channel, it is possible to classify the channels as follows.

Let us denote by $B$ the bandwidth of the transmitted signal and by $T$ its duration. Even though from a strict mathematical point of view a finite bandwidth implies an infinite duration, in practice we may assume both $T$ and $B$ to be finite. Actually, if we define $B$ and $T$ as the effective bandwidth and duration of the transmitted signals, the Heisenberg’s uncertainty principle states that the time-bandwidth product must be less than a fixed quantity:

$$BT \leq \frac{1}{2}. \quad (1.75)$$

We say that a channel is frequency-selective if its coherence bandwidth is less than the signal bandwidth, i.e. $B_c < B$. Equivalently, since $B_c$ is inversely proportional
Channel models and modes

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to \( \tau_0 \) and \( B \) is inversely proportional to \( T \), a frequency-selective channel is time-dispersive, as \( \tau_0 > T \).

Using duality arguments, we say that a channel is time-selective if its coherence time is less than the signal duration, i.e. \( T_c < T \). This implies, in the frequency domain, that \( f_{\text{max}} > B \).

We emphasize that the properties of a channel of being time and/or frequency selective is not a property of the channel per se, but it is only in relationship with the properties of the signal passing through the channel. As a further remark, we wish to anticipate that the property of being time and/or frequency selective is not necessarily a negative feature because, if properly, exploited, time and/or frequency selectivity can become the source of time and frequency diversity, as we will see in detail in the next chapters.

1.4 MIMO CHANNELS

A MIMO system with \( N_T \) transmit and \( N_R \) receive antennas is characterized by an input/output relationship

\[
y_m(t) = \sum_{n=1}^{N_T} \int_{-\infty}^{\infty} h_{mn}(t, \tau) x_n(t - \tau) d\tau, \quad m = 1, 2, \ldots, N_R,
\]

(1.76)

where \( x_n(t) \) is the signal transmitted from the \( n \)-th transmit antenna, \( y_m(t) \) is the signal received from the \( m \)-th receive antenna, and \( h_{mn}(t, \tau) \) is the impulse response of the channel between the \( n \)-th transmit and the \( m \)-th receive antennas. The system of equation in (1.76) can also be rewritten in a more compact form using a matrix notation, as

\[
y(t) = \int_{-\infty}^{\infty} H(t, \tau)x(t - \tau) d\tau,
\]

(1.77)

where \( H(t, \tau) \) is the \( N_R \times N_T \) channel matrix

\[
H(t, \tau) = \begin{pmatrix}
    h_{11}(t, \tau) & h_{12}(t, \tau) & \ldots & h_{1,N_T}(t, \tau) \\
    h_{21}(t, \tau) & h_{22}(t, \tau) & \ldots & h_{2,N_T}(t, \tau) \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{N_R1}(t, \tau) & h_{N_R2}(t, \tau) & \ldots & h_{N_RN_T}(t, \tau)
\end{pmatrix},
\]

(1.78)

\( y(t) := [y_1(t), \ldots, y_{N_R}(t)] \) is the \( n_R \)-size vector containing the signals received from the \( n_R \) antennas, whereas \( x(t) := [x_1(t), \ldots, x_{N_T}(t)] \) is the \( n_T \)-size vector containing the signals transmitted from the \( n_T \) antennas. The properties of each
impulse response $h_{mn}(t, \tau)$ can be described by the same tools introduced in the previous sections on SISO channels. What is really specific of MIMO channels is the spatial correlation among different impulse responses.

Narrowband models for the spatial correlations, based on the physical channel geometry, were proposed, for example, in [9], [10], [11]. Here, we recall briefly the space-time statistical model derived in [11]. The geometry underlying the model derivation is sketched in Figure 1.12. The model assumes that the scatterers are located along the circumference of circle of radius $R$, around the receiver. The pdf of the scatterers is modelled as a Von Mises pdf

$$p_\theta(\theta) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\theta - \mu)}, \quad \theta \in [-\pi, \pi),$$

where $I_0(\cdot)$ is the zero order modified Bessel function and $\mu$ is the mean angle of arrival. The receiver is assumed to be moving along a direction specified by the angle $\phi$. The parameter $\kappa$ controls the concentration of the scatterers around the mean angle of arrival: $\kappa = 0$ indicates isotropic scattering, whereas $\kappa = \infty$ indicates scattering concentration only on one direction. Some examples of $p_\theta(\theta)$, parameterized according to $\kappa$, are reported in Figure 1.13. The space-time correlation $R_{p,m}^{k,n}(\tau) := E\{h_{p,n}^*(t)h_{q,m}(t + \tau)\}$ assumes the form

$$R_{p,m}^{k,n}(\tau) = \int_0^{2\pi} e^{j2\pi (d_{pq} \gamma \sin \alpha \sin \theta + d_{mn} \cos(\theta - \beta))/\lambda} e^{-j2\pi D \cos(\theta - \phi)\tau} p_\theta(\theta) d\theta$$

where $D$ is the distance between the antennas, $\gamma$ is the path loss factor, $\alpha$ is the angle between the antennas, $\beta$ is the angle of arrival, $\lambda$ is the wavelength, and $\phi$ is the direction of the receiver.

Figure 1.12 Geometry of the one-ring scattering model.
where $f_D := v\lambda$. Inserting von Mises model in (1.80), it is possible to solve the integral and the result is [11]:

$$R_{p,m}^{q,n}(\tau) = \frac{e^{j\rho_{pq} \cos \alpha}}{I_0(\kappa)} I_0(\sqrt{\psi}),$$  \hspace{1cm} (1.81)

where

$$\psi := \kappa^2 - a^2 - b_{nm}^2 - c_{pq}^2 \sin^2 \alpha + 2ab_{nm} \cos(\beta - \phi) + 2c_{pq}\gamma \sin(\alpha \sin \phi - b_{nm} \sin \beta) - j2\kappa[a \cos(\mu - \phi) - b_{nm} \cos(\mu - \beta)] - c_{pq}\gamma \sin \alpha \sin \mu,$$  \hspace{1cm} (1.82)

with $a = 2\pi f_d \tau$, $b_{nm} := 2\pi d_{nm} / \lambda$ and $c_{pq} := 2\pi d_{pq} / \lambda$. Equation (1.82) refers to the diffuse component, i.e. the received components without a LOS contribution.

The simplest case of (1.82) is Clarke’s temporal correlation $J_0(2\pi f_D \tau)$, obtained for $d_{11} = \delta_{11} = 0$, i.e. single transmit-receive antenna systems and $\kappa = 0$, i.e isotropic scattering.

It is interesting to analyze the spatial correlation as a function of the element spacing. In Figure 1.14 we draw the correlation coefficient of a $2 \times 2$ MIMO system, in $\tau = 0$, as a function of the spacing between the transmit elements $\delta_{12}$ and of the

Figure 1.13 Angular distribution of scatterers according to the Von Mises pdf, for $\kappa = 0, 1, 2,$ and 4.
receive elements $d_{1:2}$. In particular, Figure 1.14a refers to channels with no line of sight (LOS), with two different parameters of the Von Mises pdf: a) refers to isotropic scattering ($\kappa = 0$); b) refers to non-isotropic scattering with $\kappa = 3$. From Figure 1.14 we observe that the correlation is in general smaller in case of non-isotropic scattering.

Possible LOS components of course play an important role in the evaluation of the correlation coefficient. As an example, Figure 1.15 refers to a channel as in Figure 1.14a), except for the presence of a LOS component, with Rice factor $K = 4$. We see from Figure 1.15 that now the correlation coefficient is much higher.

The behavior of the spatial correlation allows us to introduce the concept of coherence space. Similarly to what we did with the coherence time and bandwidth of a SISO channel, we say that the coherence space is the distance $D_c$ such that the space-time correlation (1.80), in $\tau = 0$, reaches half the value assumed in the origin of the space-time domain. In formulas, if we express the space-time correlation in (1.80) explicitly in terms the inter-element distance $d$ (assuming a uniformly spaced array), and we denote this correlation as $R_s(d, \tau)$, the space coherence is the distance $D_c$ such that $R_s(D_c, 0) = R_s(0, 0)/2$.

Virtual channel model

Let us assume now a different point of view, which can be very useful to find out the diversity order of the channel. Let us suppose that there are $L$ scatterers between the transmitter and the receiver. A possible MIMO channel model for this scenario is

$$H = \sum_{k=1}^{L} \beta_k a_R(\phi_{R,k}) a_T(\phi_{T,k}),$$

(1.83)

where $\beta_k$ is amplitude characterizing the $k$-th scatterer, whereas $a_R(\phi_{R,k})$ and $a_T(\phi_{T,k})$ are the array response vectors of the receive and transmit arrays, respectively. More specifically, let us assume to have a transmit array with $M$ uniformly spaced elements, with inter-element distance $d_T$ a receive array with $N$ uniformly spaced elements, with inter-element distance $d_R$, we have

$$a_T(\phi_{T,k}) = [1, e^{-j2\pi\theta_{T,k}}, \ldots, e^{-j2\pi(M-1)\theta_{T,k}}]^T,$$

$$a_R(\phi_{R,k}) = [1, e^{-j2\pi\theta_{R,k}}, \ldots, e^{-j2\pi(N-1)\theta_{R,k}}]^T,$$

(1.84)

where $\theta_{T,k} := d_T \sin(\phi_{T,k}/\lambda)$ and $\theta_{R,k} := d_R \sin(\phi_{R,k}/\lambda)$. This formulation is interesting because it gives a clear meaning to the rank of $H$. From (1.83) it is in fact clear that the rank of $H$ is equal to the number of scattering elements. The importance of this relationship will be clear after we will have introduced the
Figure 1.14  Spatial channel correlation coefficient for channels with no LOS; a) $\kappa = 0$, b) $\kappa = 3$. 
concept of channel mode, in the ensuing section, and we will consider in detail the capacity of MIMO channels, in Chapter 7.

1.5 MODES

One of the major advantages in a wireless communications systems based on multiple antenna transceivers is the possibility of multi-modal propagation. Different modes can in fact be used to transmit, simultaneously, more information bits: Each mode carries its own set of bits and at the receiver it is relatively easy to separate the modes and recover all the bits. This property is one of the major good features of multiple antenna systems and it motivates most of the research on this field. Nonetheless, multi-modal transmission can occur not necessarily in the spatial domain, but also in the time or polarization domain. In this section, we introduce first the concept of mode and show how modes manifest and how they can be advantageously exploited.
1.5.1 Discrete-time channel

Both block transmission through SISO systems and MIMO systems are characterized by a discrete-time I/O relationship that assumes the general form

\[ y = Hx + w, \quad (1.85) \]

where \( x \) and \( y \) are vectors of size \( M \) and \( N \), respectively, \( H \) is necessarily \( N \times M \) and \( w \) is a noise vector, of size \( N \). We do not make any assumption about neither \( M \) nor \( N \), to leave (1.85) as general as possible. Also, for the same reason, we do not impose any structure on the matrix \( H \), so that (1.85) can describe a SISO or a MIMO system.

We know, from basic properties of linear algebra, that every matrix \( H \) can be factorized by its Singular Value Decomposition (SVD):

\[ H = UV^H, \quad (1.86) \]

where \( U \) is a para-unitary matrix, with \( UU^H = I \), where \( V \) is a para-unitary matrix, with \( V^HV = I \). The matrix \( \Sigma \) may assume one of the following forms, depending on the relationship between \( N \) and \( M \):

\[ \Sigma = D, \text{if } N = M; \Sigma = \begin{pmatrix} D \\ 0 \end{pmatrix}, \text{if } N > M; \Sigma = \begin{pmatrix} D & 0 \end{pmatrix}, \text{if } N < M; \]

(1.87)

where \( D \) is a diagonal matrix of dimension \( \min(M, N) \times \min(M, N) \). The diagonal entries of \( D \) are the singular values of \( H \), whereas the columns of the matrices \( U \) and \( V \) are, respectively, left and right singular vectors of \( H \). If \( H \) is square and Hermitian, (1.86) reduces to the well known eigenvalue decomposition, where the left and right singular vectors coincide, i.e. \( U \equiv V \) and the singular values coincide with the eigenvalues and are real variables.

Considering that \( \Sigma \) is diagonal, (1.86) can also be written as

\[ H = \sum_{k=1}^{r} \sigma_k u_k v_k^H, \quad (1.88) \]

where \( r \) is the rank of \( H \) and it is equal to the number of nonnull singular values of \( H \).

This basic property is sufficient to introduce the important concept of multi-mode propagation. Let us suppose that we wish to transmit \( r \) symbols, \( s_1, s_2, \ldots, s_r \), simultaneously through the channel. We can transmit the \( M \)-size
where $v_i$ is the $i$-th channel right singular vector and $c_i$ is a coefficient which gives us the possibility to optimize our transmission system. Combining (1.85), (1.88) and (1.89), it is straightforward to check that, transmitting (1.89) produces, at the receiver, the vector

$$y = Hx = \sum_{k=1}^{r} \sum_{i=1}^{r} c_k s_k u_k^H v_i + w.$$  \hfill (1.90)

Exploiting the orthogonality of the vectors $v_i$, (1.90) becomes

$$y = \sum_{k=1}^{r} c_k s_k u_k + w.$$  \hfill (1.91)

Hence, we can recover every symbol $s_l$ by projecting $y$ along the vector $u_l$, and scaling the result, as follows

$$\hat{s}_l = \frac{1}{\sigma_l c_l} u_l^H y = s_l + w_l, \quad l = 1, \ldots, r,$$  \hfill (1.92)

where $w_l := u_l^H w / (\sigma_l c_l)$. All previous equations can be collected in a more compact matrix form, as follows. We transmit the vector

$$x = Fs = V\Phi s,$$  \hfill (1.93)

where $F := V\Phi$ is an $M \times r$ linear encoding matrix, $V$ is the matrix containing the channel right singular vectors and $\Phi$ is a diagonal matrix with diagonal entries $\Phi(i, i) = c_i$. At the receiver, we multiply the received vector $y$ times the decoding matrix $G := \Gamma U$, where $\Gamma$ is a diagonal matrix, thus obtaining

$$z = GHx = \Gamma U U^H \Sigma V^H V \Phi s + \Gamma U w = \Gamma \Sigma \Phi s + w'$$  \hfill (1.94)

where $w' := \Gamma U w$. It is important to observe the two following properties about (1.94): i) thanks to the use of channel right singular vectors at the transmitter and left singular vector at the receiver, the channel matrix is diagonalized, so that there is inter-symbol interference (ISI); ii) if the received noise is white Gaussian, i.e. the covariance matrix of $w$ is diagonal, the final noise vector $w'$ is also white and Gaussian, so that symbol-by-symbol decision is optimal. This property can be proved immediately. In fact, if denote by $C_w = \sigma_w^2 I$ the covariance matrix of $w$,
the covariance matrix of $w$ is

$$C_w := E\{w'w'^H\} = \sigma_n^2 UU^H \Gamma \Gamma^H = \sigma_n^2 \Gamma \Gamma^H$$  \hspace{1cm} (1.95)$$

and this matrix is certainly diagonal.

We say that every pair of vectors $(u_k, v_k)$ constitutes a channel mode. Transmitting using the scheme of (1.89) is equivalent to transmit every symbol through one channel mode. From this general set-up, it is clear that the maximum number of symbols that can be transmitted simultaneously is equal to the rank of the channel matrix $H$. Depending on the structure of $H$, we may talk of modes in time, space or polarization domain.

### 1.5.2 Continuous-time channel

The discrete-time formulation presented in the previous section has been useful to introduce the concept of mode. However, in the discrete-time formulation there is no immediate link between the modes and the physical channel characteristics. Finding such a relationship is indeed not an easy task, in general. Nevertheless, in some cases of practical interest, it is possible to establish such a relationship. This happens, for example, for underspread SISO channels. It is the scope of this section to give an analytic, albeit approximate, expression for the eigenfunctions of underspread SISO channels, useful to shed light on the physical meaning of mode.

We consider the case where the channel impulse response is square integrable, i.e.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(t, \tau)|^2 dt d\tau < \infty. \hspace{1cm} (1.96)$$

In practice, there are at least three cases where $h(t, \tau)$ does not respect (1.96): i) LTI channels, where $h(t, \tau)$ is constant along $t$; ii) multiplicative channels, where $h(t, \tau) = m(t) \delta(\tau - \tau_0)$; and iii) multipath channels with specular reflections, where $h(t, \tau)$ contains Dirac pulses. We consider such situations as limiting cases and we will show that indeed the results for such cases can be described as limiting values of the expressions derived for the finite norm case. Nevertheless, it is worth pointing out that, in case of LTI or multiplicative channels, the system eigenfunctions are known in closed form, exactly, so that in such a case, there is no real problem.

If $h(t, \tau)$ satisfies (1.96), there exists a countable series of real numbers $\lambda_i$ and two sets of orthonormal functions $u_i(t)$ and $v_i(t)$ such that the following pair
of integral equations hold true 2:

\[ \lambda_i u_i(t) = \int_{-\infty}^{\infty} h(t, t - \tau)v_i(\tau)d\tau \quad (1.97) \]

and

\[ \lambda_i v_i(\tau) = \int_{-\infty}^{\infty} h^*(t, t - \tau)u_i(t)dt. \quad (1.98) \]

The scalar values \( \lambda_i \) are the so-called singular values of the system, whereas the functions \( u_i(t) \) and \( v_i(t) \) are known as the left and right singular functions, associated to \( \lambda_i \). Equations (1.97) and (1.98) are the continuous-time counterpart of the SVD (1.86).

Inserting (1.97) in (1.98), we also get

\[ \lambda_i^2 v_i(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^*(t, t - \tau)h(t, t - \theta)v_i(\theta)d\theta dt \quad (1.99) \]

so that \( v_i(\tau) \) is the eigenfunction of the system

\[ \lambda_i^2 v_i(\tau) = \int_{-\infty}^{\infty} \tilde{h}(\tau, \theta)v_i(\theta)d\theta, \quad (1.100) \]

whose kernel is

\[ \tilde{h}(\tau, \theta) := \int_{-\infty}^{\infty} h^*(t, t - \tau)h(t, t - \theta)dt. \quad (1.101) \]

Similarly, substituting (1.98) in (1.97), \( u_i(t) \) is an eigenfunction of the system with kernel

\[ h(t, \theta) := \int_{-\infty}^{\infty} h(t, t - \tau)h^*(\theta, \theta - \tau)d\tau. \quad (1.102) \]

In Appendix B, we review the basic theory of slowly-varying linear Hermitian operators, useful to find an approximate expression for the eigenfunctions of underspread channels. However, before recalling the major results of such a theory, it is useful to anticipate a few simple derivations that provide a first order approximate solution.

The most known example where the eigenfunctions are known exactly is the case of linear time-invariant channels. In such a case, \( h(t, \tau) = h(\tau) \) and the channel eigenfunctions are \( u_i(t) \equiv v_i(t) = e^{j2\pi f_i t}, \) whereas the eigenvalues are \( \lambda_i = H(f_i), \) where \( H(f) \) is the channel transfer function. This property can be checked immediately by simply substituting these expressions for \( u_i(t) \) and \( v_i(t) \) in

2 There is no loss of generality in assuming that \( \lambda \) is real as every phase of \( \lambda \) could be incorporated n the channel singular functions.
(1.97) or (1.98). Let us now see how to extend the solution to the case of underspread channels. From what we have seen in Section 1.3.1.1, a channel is underspread if the support of its spread function is very limited in delay, Doppler or both. Let us consider, for example, the case where the support along \( \tau \) is very limited. We search for a solution of the system (1.97) within the class of functions having the following form

\[ v_i(t) = A_i e^{j\phi_i(t)}, \quad t \in I_i, \]  

(1.103)

where the function \( \phi_i(t) \) and the interval \( I_i \) have to be determined. The amplitude \( A_i \) must insure that \( v_i(t) \) has unit norm.

Inserting (1.103) in (1.97), we get

\[ \lambda_i u_i(t) = \int_{-\infty}^{\infty} h(t, \tau) e^{j\phi_i(t-\tau)} d\tau. \]  

(1.104)

Since the channel is assumed to have a very limited support along \( \tau \), we can expand \( \phi_i(t-\tau) \) around \( t \) and keep only the first two terms

\[ \phi_i(t-\tau) \approx \phi_i(t) - \tau \dot{\phi}_i(t), \]  

(1.105)

where \( \dot{\phi}_i(t) \) denotes the first order time derivative of \( \phi_i(t) \). Substituting (1.105) in (1.104), we get

\[ \lambda_i u_i(t) \approx A_i e^{j\phi_i(t)} \int_{-\infty}^{\infty} h(t, \tau) e^{-j\tau \dot{\phi}_i(t)} d\tau. \]  

(1.106)

Recalling (1.11), we may rewrite (1.106) as

\[ \lambda_i u_i(t) \approx A_i e^{j\phi_i(t)} H \left( t, \frac{\dot{\phi}_i(t)}{2\pi} \right) = v_i(t) H \left( t, \frac{\dot{\phi}_i(t)}{2\pi} \right). \]  

(1.107)

This relationship implies that, if \( \phi(t) \) is such that \( H(t, f) \) is constant along the curve \( f = \phi_i(t)/2\pi \), then \( v_i(t) \), as expressed in (1.103) is a possible solution of (1.97), with

\[ \lambda_i = H(t, \phi_i(t)/2\pi), \]  

(1.108)

where \( I_i \) is the time interval where (1.108) admits a solution, and \( u_i(t) = v_i(t) \). In general, (1.108) may admit more than one solution, so that the most general form for the solution is

\[ v_i(t) = \sum_{k=1}^{n_i(t)} A_{i,k} e^{j\phi_{i,k}(t)}, \quad t \in I_i, \]  

(1.109)
where the instantaneous phases $\phi_{i,k}(t)$ are all such that the corresponding instantaneous frequency $f_{i,k}(t) := \phi_{i,k}(t)/2\pi$ are all solution of the implicit equation (1.108); $n_i$ is the number of solutions of (1.108), as a function of $t$.

Therefore, in summary, we can state that a first order solution for the eigenfunction problem is given by functions $v_i(t)$ composed of components having constant amplitude and instantaneous frequencies given by the curves, in the time-frequency domain, where $H(t,f)$ is constant and equal to $\lambda_i$, the singular value whom $v_i(t)$ refers to.

The previous derivations started assuming that $S(\nu,\tau)$ or, equivalently, $h(t,\tau)$, has a very limited support along $\tau$. If we consider now the case where $S(\nu,\tau)$ has very limited support along $\nu$, using the duality property of the Fourier Transform, we may follow similar arguments as before, but in the frequency domain, to show that an approximate model for $v_i(t)$ is given by those functions whose Fourier Transform $V_i(f)$ assumes the form

$$V_i(f) = A_i e^{j\Phi_i(f)}, \quad t \in I_i,$$  

(1.110)

where $\Phi_i(f)$ is such that

$$\lambda_i = H\left(\frac{\phi_i(f)}{2\pi}, f\right).$$  

(1.111)

The above derivations are clearly approximate. Since the kernel (1.101) is Hermitian, we can exploit the general results derived in Appendix B to state that the singular function $v_i(t)$ assumes the following form

$$v_i(t) = \sum_m a_{i,m}(t)e^{j\phi_{i,m}(t)},$$  

(1.112)

where the instantaneous phase $\phi_{i,m}(t)$ of each component is such that the corresponding instantaneous frequency $f_{i,m}(t) := \phi_{i,m}(t)/2\pi$ is a solution of

$$|H(t,f_{i,m}(t))|^2 = \lambda_i^2,$$  

(1.113)

whereas the instantaneous envelope is

$$a_{i,m}(t) = \frac{1}{\sqrt{\frac{|H(t,f)|^2}{\partial f}}_{f=\phi_{i,m}(t)/2\pi}}.$$

(1.114)

In words, $v_i(t)$ is a multi-component signal whose components have an instantaneous frequency given by the curve, in the time-frequency domain $(t,f)$, where the modulus of the channel time-varying transfer function is constant and equal to $\lambda_i^2$. 
In words, \( v_i(t) \) is a multi-component signal whose components have an instantaneous frequency given by the curve, in the time-frequency domain \((t, f)\), where the modulus of the channel time-varying transfer function is constant and equal to \( \lambda_i^2 \).

Invoking duality arguments, we can also state that if the channel spread along the Doppler axis is small, the function \( v_i(t) \) is such that its spectrum \( V_i(f) \) assumes the form

\[
V_i(f) = \sum_m A_{i,m}(f)e^{j\Phi_{i,m}(f)},
\]

where the instantaneous phase \( \Phi_{i,m}(f) \) of each spectrum component is such that the corresponding group delay \( t_{i,m}(t) := -\Phi_{i,m}(f)/2\pi \) is a solution of

\[
|H(t_{i,m}(f), f)|^2 = \lambda_i^2,
\]

whereas the instantaneous spectrum envelope of each component is

\[
A_{i,m}(f) = \left[ \frac{1}{\sqrt{\left|\frac{\partial|H(t,f)|^2}{\partial f}\right|}} \right]_{t=-\Phi_{i,m}(f)/2\pi}.
\]

Not all values of \( \lambda_i \) are admissible. From (1.113), evidently \( \lambda_i \) must belong to the interval

\[
\min_{t,f}|H(t,f)| \leq \lambda_i \leq \max_{t,f}|H(t,f)|.
\]

But even within this interval, not all values are admissible. It is shown in Appendix B that the only possible values satisfy the so called area rule, which can be explained graphically, with the help of Figure 1.16. In Figure 1.16 a possible contour plot of the square modulus of the channel transfer function is sketched. Given the relationship between instantaneous phase and frequency, if we consider the lower curve \( f_{k,1}(\mu) \), we can write

\[
\phi_k(t_2) = \phi_k(t_1) + 2\pi \int_{t_1}^{t_2} f_{k,1}(\theta) d\theta.
\]

On the other hand, if we consider the upper curve, instead, we have

\[
\phi_k(t_1) = \phi_k(t_2) + 2\pi \int_{t_2}^{t_1} f_{k,2}(\theta) d\theta.
\]

Combining these two equations and considering that any phase value gives rise to the same signal value if an integer multiple of \( 2\pi \) is added to the phase, we can write

\[
\int_{t_1}^{t_2} [f_{k,2}(\theta) - f_{k,1}(\theta)] d\theta = k.
\]
where $k$ is an integer number. Equation (1.121) states that the only levels of $|H(t, f)|$ that give rise to an eigenvalue $\lambda_i$ are the ones such that the area within the two instantaneous frequency curves (the dashed area in Figure 1.16) is a positive integer number.

After having derived the approximate analytic model (1.112), with (1.113), we also need to check that the class of functions $v_i(t)$ and $u_i(t)$ constitute two classes of orthonormal functions. To prove this property, it is useful to resort to time-frequency distributions (TFD). A few properties about TFD are recalled in Appendix C. In particular, we recall the Wigner-Ville Distribution (WVD), as it plays a central role in the theory of TFD’s. The WVD $W_s(t, f)$ of a signal $s(t)$ is defined as

$$W_s(t, f) = \int_{-\infty}^{\infty} s^*(t - \tau/2)s(t + \tau/2)e^{-j2\pi f \tau} d\tau.$$  

One of the main properties of the $W_s(t, f)$, useful for our analysis here, is that it is maximally concentrated, in the time-frequency domain, along the instantaneous frequency of the signal. The degree of concentration depends on the amplitude modulation of the signal: The smoother is the amplitude modulation, the more concentrated is the WVD. This property is particularly useful in the analysis of underspread channels, because the amplitude modulation given by (1.114) is indeed a smooth function. The other important property to be used is the so called Moyal’s formula, see, e.g. (1.183), stating that the scalar product between the WVD’s of two signals $x(t)$ and $y(t)$ is equal to the square modulus of the scalar product between the two signals. Therefore, we can check the orthogonality between two functions, looking at the scalar product of their WVD. Any two singular functions $v_i(t)$ and $v_j(t)$, $i \neq j$, relative to different singular values, have instantaneous frequencies located, by construction, over non-overlapping curves of the time-frequency plane.
This happens because those instantaneous frequencies refer to different contour levels of $|H(t, f)|^2$. Therefore, if the WVD are also maximally concentrated along those instantaneous frequencies, the scalar product between the WVD’s of $v_i(t)$ and $v_j(t)$ is null. Hence, the singular function associated to different eigenvalues are indeed orthogonal.

1.5.3 Examples

A few examples are useful to understand the meaning of the derivations illustrated in the previous section.

1.5.3.1 LTI channels

A linear time invariant (LTI) channel is a channel whose impulse response $h(t, \tau)$ does not depend on $t$, i.e.

$$h(t, \tau) = h(\tau).$$

Equivalently, its transfer function depends only on $f$, i.e. $H(t, f) = H(f)$. As a consequence, the curve, in the time-frequency domain where $|H(t, f)|$ is constant is given by lines of equation $f = f_i$. This behavior of the instantaneous frequency characterizes a signal having only one frequency, constant for all times. This signal is clearly a complex exponential of the form $\exp(j2\pi f_0 t)$. And in fact, this is a well known property, which can be verified immediately by plugging $x(t) = \exp(j2\pi f_0 t)$ in (1.10) to check that the corresponding output is $y(t) = H(f_i)x(t)$.

1.5.3.2 Multiplicative channels

A multiplicative channel is characterized by the following input-output relationship

$$y(t) = m(t)x(t)$$

(1.123)

or, incorporating a possible delay, $y(t) = m(t)x(t - \tau_0)$. The impulse response of such a channel is

$$h(t, \tau) = \delta(\tau - \tau_0)m(t).$$

As an example, the multipath model (1.16) degenerates into a multiplicative channel when the delays are all equal to each other, e.g. $\tau_k = \tau_0, \forall k$. In such a case,

$$h(t, \tau) = \delta(\tau - \tau_0) \sum_{q=0}^{Q-1} h_q e^{j2\pi f_q t}. \quad (1.124)$$

The time-varying transfer function of a multiplicative channel is $H(t, f) = e^{-j2\pi \tau_0 f} m(t)$ and its modulus $|H(t, f)| = |m(t)|$ is constant over lines of equation $t = t_0$. Since $|H(t, f)|$ is constant along $f$, according to (1.115) and (1.116), the
group delay is constant and thus the spectrum of the channel eigenfunctions has constant amplitude and linear phase. Hence, the eigenfunctions are Dirac pulses. In fact, if the input is \( x(t) = \delta(t - \tau_0) \), the corresponding output \( y(t) \) is proportional to the input and it is \( y(t) = m(\tau_0)\delta(t - \tau_0) = m(\tau_0)x(t) \).

Before concluding this section, we wish to remark that even though the LTI and multiplicative channels are characterized by impulse responses which are not square-integrable, the models (1.112) and (1.115) in these two cases hold exactly.

1.5.3.3 Two-ray multipath channel

Let us consider a multipath channel composed of two rays, i.e. (1.16) with \( Q = 2 \). We assume that the delays \( \tau_k \) are not equal to each other (as this would be a particular case of the multiplicative channel analyzed before). In such a case, the channel singular function must be a solution of

\[
\lambda_i e^{j\psi_i} v_1(t - t_d)e^{j2\pi f_0 t} = h_0 v_i(t - \tau_0)e^{j2\pi f_0 t} + h_1 v_i(t - \tau_1)e^{j2\pi f_1 t}.
\]

(1.125)

Setting \( t_d = \tau_0 \) and \( f_d = f_0 \), we may rewrite (1.125) as

\[
\lambda_i v_i(t - \tau_0) = h_0 v_i(t - \tau_0) + h_1 v_i(t - \tau_1)e^{j2\pi(f_1 - f_0)t},
\]

(1.126)

Setting \( \theta = t - \tau_0 \), we have

\[
\lambda_i v_i(\theta) = h_0 v_i(\theta) + h_1 v_i(\theta - \Delta\tau)e^{j2\pi\Delta f(\theta + \tau_0)},
\]

(1.127)

where \( \Delta f := f_1 - f_0 \) and \( \Delta\tau := \tau_1 - \tau_0 \). It is straightforward to verify, by direct substitution, that the solution of (1.127) is given by the functions

\[
v_i(t) = e^{j2\pi \frac{\tau_1}{\tau_0}} e^{j\frac{\Delta f}{\Delta\tau} t^2}
\]

(1.128)

parameterized with respect to the variable \( \alpha \), which is related to \( \lambda_i \) by

\[
\lambda_i e^{j\psi_0} = h_0 + h_1 e^{j2\pi\alpha} e^{-j\pi\Delta f(\tau_1 + \tau_0)},
\]

(1.129)

where the phase \( \psi_0 \) is chosen in order to have a real non-negative value for \( \lambda_i \). Therefore, the singular functions of 2-rays channels are *chirp* signals, i.e. signals whose instantaneous frequency is a linear function of time and, in particular, whose sweep rate is \( \Delta f/\Delta\tau \). Furthermore, the eigenvalues are related to the channel parameters by (1.129). We can now verify the validity of (1.112) and (1.113) in this case. In fact, the channel transfer function is

\[
H(t, f) = h_0 e^{j2\pi(f_0 t - f\tau_0)} + h_1 e^{j2\pi(f_1 t - f\tau_1)} = e^{j2\pi(f_0 + f_1 - f\tau_0)(h_0 + h_1 e^{j2\pi(\Delta\tau - \Delta f)})}.
\]

(1.130)
where the parameter $\alpha$ is related to $\lambda_i$ by

$$\lambda_i e^{j\varphi_0} = h_0 + h_1 e^{j2\pi \alpha},$$

(1.132)

where $\varphi_0$ is any constant phase. Solving (1.131) for $f$ and using (1.112), the instantaneous frequency of the channel eigenfunctions components is

$$f = f_i(t) = \frac{\Delta f}{\Delta \tau} t + \frac{\alpha}{\Delta \tau},$$

(1.133)

so that, according to (1.112), the channel eigenfunctions are

$$v_i(t) = e^{j2\pi \frac{\Delta f}{\Delta \tau} t} e^{j\frac{\alpha}{\Delta \tau} t^2}.$$  

(1.134)

Notice that this expression coincides with (1.128). Furthermore, if the channel is underspread, i.e. $(\tau_1 + \tau_0)\Delta f \ll 1$, the channel eigenvalue obtained through (1.129) or (1.132) coincide by simply setting $\varphi_0 = \psi_0$.

1.5.3.4 General multipath channel

To check the validity of the analytic model in the general case, it is useful to analyze the channel eigenfunctions in the joint time-frequency domain, rather than simply in the time or in the frequency domain. The basic tool for verifying the analytic model is given by time-frequency distributions (TFD).

More specifically, we checked the validity of (1.112) or (1.115) using the following scheme. Given the impulse response $h(t, \tau)$ of the CT system, i) we build the channel matrix $H(n)$ of the equivalent DT system, using (1.41); ii) we compute the SVD of $H$; iii) we compute the WVD of the right and left singular vectors associated to the generic singular value $\lambda_i$; and iv) we compare the energy distribution of these TFD with the contour plot of $|H(t, f)|$ corresponding to level $\lambda_i$. We used as a basic tool to analyze the signals in the time-frequency domain the Smoothed Pseudo-Wigner-Ville Distribution (SPWVD) with reassignment, introduced in [12], for its property of having low cross terms without degrading the resolution. We considered as a test system a communication channel affected by multipath propagation, thus described by the CT impulse response

$$h(t, \tau) = \sum_{q=0}^{Q-1} h_q e^{j2\pi f_q \tau} \delta(t - \tau_q),$$

where each path is characterized by the triplet of amplitude $h_q$, delay $\tau_q$ and Doppler shift $f_q$. We generated the amplitudes $h_q$ as independent identically distributed (iid) complex Gaussian random variables with zero mean and unit variance (the Rayleigh fading model), and the variables $\tau_q$ and $f_q$ as iid random variables with
Figure 1.17 Comparison between contour lines of $|H(t, f)|$ and TFD’s of channel singular vectors - a) $|H(t, f)|$; b) contour lines of $|H(t, f)|$ corresponding to levels $\lambda_{16}$ (dashed line) and $\lambda_{32}$ (solid line); c) SPWVD of $v_{16}$; d) SPWVD of $v_{32}$.
uniform distribution within the intervals $[0, \Delta \tau]$ and $[-\Delta f/2, \Delta f/2]$, respectively. An example, relative to a multipath channel composed of $Q = 12$ paths, with $\Delta \tau = 4T_s$ and $\Delta f = 4/N T_s$, $N = 128$, is reported in Figure 1.17, where we show: a) the mesh plot of $|H(t, f)|$, b) two contour plots of $|H(t, f)|$ corresponding to the levels $\lambda_{16}$ (dashed line) and $\lambda_{32}$ (solid line); c) the contour plot of the SPWVD of $v_{16}$; d) the contour plot of the SPWVD of $v_{32}$.

It is worth noticing how, in spite of the rather peculiar structure of the contour plots of $|H(t, f)|$, the SPWVD’s of the two singular functions are strongly concentrated along curves coinciding with the contour lines of $|H(t, f)|$ corresponding to the associated singular values, as predicted by the theory.

It is also interesting to observe the bubble-like structure of the two SPWVD’s. Indeed this behavior is quite common, because in general the contour lines of the time-varying transfer function are typically closed curves.

The singular functions $u_i(t)$ and $v_i(t)$ are the continuous time channel modes. They play, in fact, a similar role of the left and right singular vectors of the channel matrix, in the discrete-time domain. In fact, the optimal strategy for transmitting a set of symbols $s_i[k] := s[iK + k]$, $k = 0, \ldots, K - 1$, in the presence of additive white Gaussian noise (AWGN), is to send the signal

$$x(t) = \sum_{k=0}^{K-1} c_k s[k] v_k(t)$$

where $v_k(t)$ is the right singular function associated to the $k$-th eigenvalue of the channel response $h(t, \tau)$ and $c_k$ are coefficients used to allocate the available power among the transmitted symbols according to some optimization criterion [13]. Using (1.97), the received signal is thus

$$y(t) = \int_{-\infty}^{\infty} h(t, \tau) x(t - \tau) d\tau + w(t) = \sum_k c_k \lambda_k s[k] u_k(t) + w(t), \quad (1.136)$$

where $u_k(t)$ is the left singular function associated to the $k$-th singular value of $h(t, \tau)$ and $w(t)$ is AWGN. Hence, by exploiting the orthonormality of the functions $u_k(t)$, the transmitted symbols can be estimated by simply taking the scalar products of $y(t)$ with the left singular functions, i.e.

$$\hat{s}[m] = \frac{1}{\lambda_m c_m} \int_{-\infty}^{\infty} y(t) u_m^*(t) dt = s[m] + w[m], \quad (1.137)$$

where the noise samples sequence $w[m] := \int_{-\infty}^{\infty} w(t) u_m^*(t) dt$ constitutes a sequence of iid Gaussian random variables.
Figure 1.18 Analysis of a real channel; co-polarized channels; mobile receiver: a) Modulus of $H(t, f)$; b) Modulus of $S(\nu, \tau)$. 
Figure 1.19  Analysis of a real channel; co-polarized channels; mobile receiver: a) Modulus of $H(t, f)$; b) Modulus of $S(\nu, \tau)$. 
1.6 ANALYSIS OF REAL DATA

In this section, we show some real MIMO channel measurements gathered at the University of Bristol, from the group coordinated by Prof. M. Beach, within the project denominated Smart Antenna Technology in Universal bRoadband Networks (SATURN), supported by the European Union, within the fifth Information Society Technology (IST) Program Framework. The experimental setup is as follows. The transmitter has four transmit antennas, each antenna with dual polarization. The transmit antennas were spaced apart by $20\lambda$. The transmitted signal occupies a bandwidth of 20 MHz, from 1920 to 1930 MHz. The element spacing at the receiver is $0.5\lambda$. Several sets of data have been gathered, in different conditions. We show some results concerning two scenarios: (A) time-varying channel, where the receiver is moving; and (B) stationary channel, where the receiver is fixed.

In Figure 1.18 we show the modulus of the time-varying transfer function and of the corresponding delay-Doppler spread function of a channel under scenario (A). The data reported in Figure 1.18 refer to a time observation window of 0.8 sec. It is evident from Figure 1.18 that the spread function has a nonnull support and thus the channel is really time-variant.

As an example of time-invariant channel, in Figure 1.19 we report the behaviors of, again, $|H(t, f)|$ and $|S(\nu, \tau)|$, taken as in Figure 1.18, except that now the receiver is fixed and there are no moving objects around the receiver. We can see from Figure 1.19 that now the spread function is highly concentrated around the origin and in fact the transfer function is practically constant over time, whereas it varies along the frequency, because of multipath propagation.

1.7 SUMMARY

In this chapter, we have introduced the main parameters characterizing a wireless channel. We have introduced the concepts of time, bandwidth and space coherence, which will play a fundamental role in devising the proper transmission strategies, as we will show in the next chapters, able to catch all the available diversity offered by a MIMO channel. Basic references on time-varying channel models are, for example, [14], [16], and [15]. A more in depth analysis of MIMO channels is given in the two recent books [3] and [17] and in journal papers, like, e.g., [11], [9], [8], [10], [18] and [19].

References

Channel models and modes


APPENDIX 1A.1: EIGENFUNCTIONS OF SLOWLY VARYING LINEAR OPERATORS

A linear operator is a rule for mapping an “input” signal \( a(x) \) onto an “output” signal \( b(x) \), which respects the superposition principle. We denote this mapping through the symbol \( b(y) = L[a(x)] \). The superposition principle states that if we apply the operator to a linear combination of functions, namely to \( a(x) = \sum_{m=1}^{M} c_m a_m(x) \), the result is the superposition, or linear combination, of the partial results \( b_k(y) = L[a_k(x)] \) through the same combination coefficients. In formulas, if \( a(x) = \sum_{m=1}^{M} c_m a_m(x) \) and \( b_m(y) = L[a_m(x)] \), then \( b(y) = L[a(x)] = \sum_{m=1}^{M} c_m L[a_m(x)] \).

Every linear operator is characterized by an input/output relationship which can be expressed through the following integral equation

\[
b(x) = \int_{-\infty}^{\infty} K(x, y) a(y) dy,
\]

where \( K(x, y) \) is known as the system kernel and describes the linear operator completely. The variables \( x \) and \( y \) that we use here may have different physical meaning, as they may denote spatial coordinates or temporal variables, for example. In (1.138) we have assumed the variables \( x \) and \( y \) to be scalar (or mono-dimensional) variables. However, (1.138) can be generalized to the multi-dimensional case, as follows

\[
b(x) = \int_{-\infty}^{\infty} K(x, y) a(y) dy,
\]

where now \( x \) and \( y \) are multi-dimensional vectors. In such a case, \( K(x, y) \) is often referred to as the Green’s function.

In this section we do not specify the physical meaning of the variables \( x \) and \( y \) to derive general results which can be applied to a variety of situations. In practice, \( x \) and \( y \) may indicate temporal quantities (and in that case they would be scalar variables), or spatial variables, or even joint space-time coordinates. In the following, we will use a terminology related to spatial coordinates, but the same considerations apply equally well to temporal coordinates. We will also consider only the scalar case, because in such a case it is possible to derive some important results in closed form and this is useful to better understand the main properties of linear operators.
Specifically, we will concentrate on some functions that play a special role in the study of linear operators, namely the eigenfunctions or singular functions. If the operator has finite norm, i.e.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(x, y)|^2 \, dt \, d\tau < \infty,$$

then there exists a countable set of singular values $\lambda_i$ and two classes of orthonormal functions $v_i(x)$ and $u_i(x)$, named right and left singular functions, such that the following system of integral equations holds true

$$\lambda_i u_i(y) = \int_{-\infty}^{\infty} K(x, y) v_i(x) \, dx,$$  \hspace{1cm} (1.141)

$$\lambda_i v_i(x) = \int_{-\infty}^{\infty} K^*(x, y) u_i(y) \, dy.$$  \hspace{1cm} (1.142)

In particular, if the Kernel is Hermitian, i.e. if $K(y, x) = K^*(x, y)$, the singular functions $u_i(x)$ and $v_i(x)$ coincide and we may talk, properly, of eigenfunctions.

Hence, a function $\psi_i(x)$ is an eigenfunction of the operator having kernel $K(x, y)$, associated to the eigenvalue $\lambda_i$, if $\psi_i(x)$ is solution of the following integral equation

$$\lambda_i \psi_i(x) = \int_{-\infty}^{\infty} K(x, y) \psi_i(y) \, dy.$$  \hspace{1cm} (1.143)

This means that the effect of applying an operator on one of its eigenfunctions consists simply in the multiplication of the eigenfunction by a constant. In general, since the eigenfunctions are solution of (1.143), the eigenfunctions have a structure that depends on the system kernel. The only exceptions of this statement are given by two special classes of systems: i) the space-invariant systems; and the multiplicative systems.

**HOMOGENEOUS SYSTEMS**

A homogeneous, or space-invariant, system is a system whose kernel satisfies the following property

$$K(x, y) = K(x - y).$$  \hspace{1cm} (1.144)

The structure of the eigenfunctions of this kind of operators is well known. It is straightforward to verify, in fact, that the eigenfunctions of such operators are the exponential functions $\psi(x) = e^{jpx}$. In fact, substituting such an expression into
(1.144), we get
\[ \int_{-\infty}^{\infty} K(x-y)e^{jpy}dy = \int_{-\infty}^{\infty} K(y)e^{jpy}du = e^{jpx} \int_{-\infty}^{\infty} K(u)e^{-jpu}du = \lambda \psi(x) \] (1.145)
where
\[ \lambda := \int_{-\infty}^{\infty} K(u)e^{-jpu}du. \] (1.146)

It is important to point out that exponential functions of the form \( e^{jpx} \) are eigenfunctions of every linear homogeneous system, irrespective of the specific structure of the system or, in other words, of the system kernel function.

**MULTIPlicative systems**

A multiplicative system is characterized by a kernel having the form \( K(x, y) = K(x)\delta(y - x) \), where \( \delta(x) \) denotes the Dirac pulse. The input/output relationship of such systems is
\[ b(x) = \int_{-\infty}^{\infty} K(x)\delta(y - x)a(y)dy = K(x)a(x). \] (1.147)

This relationship shows that the action of such a system on the input signals is simply to multiply the input by a function \( K(x) \). It is easy to verify that the eigenfunctions of such systems are the Delta functions. In fact, if \( a(x) = \delta(x - x_0) \), we have
\[ b(x) = \int_{-\infty}^{\infty} K(x)\delta(y - x)\delta(y - x_0)dy = K(x_0)\delta(x - x_0) = K(x_0)a(x). \] (1.148)

Once again, the Dirac functions are eigenfunctions of every multiplicative system.

**Weakly homogeneous systems**

If the operator is neither homogeneous nor multiplicative, it is much more difficult to derive an analytic expression of its eigenfunctions and, in general, there is no such closed form expression. Nonetheless, since in many practical applications, we have operators that depart only slightly from the homogeneous condition, it is useful to derive a closed form expression, although approximate, for the eigenfunctions of weakly inhomogeneous (or slowly time-varying) operators. We start by using a
change of variable which enables us to write the kernel $K(x, y)$ as follows

$$K(x, y) = K(x - y, \frac{\epsilon}{2}(x + y)),$$

(1.149)

where $\epsilon$ is an instrumental parameter that measures the departure from homogeneity. When $\epsilon = 0$, the system is perfectly homogeneous, whereas a small value of $\epsilon$ denotes weak inhomogeneity. In the following, we will review the perturbation method illustrated in [20] to derive an approximate expression for the eigenfunctions of weakly inhomogeneous operators, valid for small values of $\epsilon$.

We start considering the Taylor series expansion of the kernel around the point $\epsilon = 0$:

$$K(x - y, \frac{\epsilon}{2}(x + y)) = K_0(x - y) + \frac{\epsilon}{2}(x + y)K_1(x - y) + \cdots$$

(1.150)

Similarly, we make explicit the dependence of the eigenfunctions on the parameter $\epsilon$ and, for small $\epsilon$, we consider the Taylor’s series development

$$\psi(x, \epsilon) = \psi_0(x) + \epsilon\psi_1(x) + \cdots,$$

(1.151)

where $\psi_k(x) := \frac{\partial^k \psi(x)}{\partial x^k}$. Hence, we wish to solve the following system

$$\lambda \psi(x, \epsilon) = \int_{-\infty}^{\infty} K(x - y, \frac{\epsilon}{2}(x + y))\psi(y, \epsilon)dy.$$  

(1.152)

In particular, we search for a solution expressed in the analytic form

$$\psi_k(x, \epsilon) = A(x, \epsilon)e^{i\phi(x, \epsilon)},$$

where $A(x)$ and $\phi(x)$ represent the amplitude and phase of $\psi(x)$.

Following the same approach as in [20], we introduce the parameter $q = x\epsilon$ and consider the Taylor’s series expansion of $K(x - y, \frac{\epsilon}{2}(x + y))$ along its second variable, in the neighborhood of $y = x$, i.e.

$$K(x - y, \frac{\epsilon}{2}(x + y)) = K_0(x - y, q) - \frac{\epsilon}{2}(x - y)K_{01}(x - y, q) + \frac{\epsilon^2}{8}(x - y)K_{02}(x - y, q) + \cdots$$

(1.153)

where we have introduced the symbol

$$K_{lm}(p, q) := \frac{\partial^{l+m} K(p, q)}{\partial p^l q^m}.$$
We expand, similarly, $\psi(x, \epsilon)$. In this case, however, we have to be careful because, in general, small variations of the phase $\phi(x; \epsilon)$ have a greater impact on the integral (1.152) than corresponding variations of the amplitude $A(x; \epsilon)$. To take into account this different sensitivity, we express the Taylor’s series expansion of the $\psi(y; \epsilon)$ in the neighborhood of $y = x$ as follows

$$\psi(ey) = e^{i(\sum_{n=0}^{\infty}(-1)^n e^\epsilon(x-y)^n \phi^n(q)/(n!))} \sum_{n=0}^{\infty} (-1)^n e^n A_n(q)(x-y)^n. \quad (1.154)$$

In this way, the first order expansion, as a function of $\epsilon$, is

$$\psi(ey) \approx e^{i \phi(q)} e^{-j(x-y)\phi'(q)} A_0(q), \quad (1.155)$$

i.e. it retains two terms of the phase and one term of the amplitude. Similarly, the second order expansion retains three terms of the phase and two terms for the amplitude, i.e.

$$\psi(ey) = e^{i \phi(q)} e^{-j(x-y)\phi'(q)} - e^\epsilon (x-y) A_0(q) - \epsilon (x-y) A_1(q)) \approx e^{i \phi(q)} e^{-j(x-y)\phi'(q)} [1 + j \epsilon (x-y)^2 \phi''(q)/2] [A_0(q) - \epsilon (x-y) A_1(q)]$$

$$\approx e^{i \phi(q)} e^{-j(x-y)\phi'(q)} A_0(q) + e^{i \phi(q)} e^{-j(x-y)\phi'(q)} (x-y) A_1(q)$$

$$+ j (x-y) \phi''(q)/2 A_0(q) - A_1(q) \epsilon.$$

(1.156)

1.7.0.5 First order solution

We derive now the first order solution of (1.152). We consider the following expansions

$$K_0(x-y, \frac{1}{2}(x+y)) \approx K_0(x - y, q);$$

$$\psi(ey) \approx e^{i \phi(q)} e^{-j(x-y)\phi'(q)} A_0(q). \quad (1.157)$$

Introducing (1.157) in (1.152) and setting $x - y = u$, we get

$$A_0(q) e^{i \phi(q)} \int_{-\infty}^{\infty} K_0(u, q) e^{-j u \phi'(q)} du = \lambda A_0(q) e^{i \phi(q)}. \quad (1.158)$$

This equation is satisfied if $\phi(q)$ is solution of the following implicit equation

$$\int_{-\infty}^{\infty} K_0(u, q) e^{-j u \phi'(q)} du = \lambda$$

(1.159)
Hence, introducing the function
\[
\tilde{K}(p, q) := \int_{-\infty}^{\infty} K_0(u, q)e^{-jup}du,
\]
(1.160)
\(\phi(q)\) is, equivalently, solution of
\[
\tilde{K}(\phi'(q), q) = \lambda.
\]
(1.161)

This means that \(\phi'(q)\) describes the curve, in the \((u, q)\) plane, where the function \(\tilde{K}(u, q)\) is constant and equal to \(\lambda\). In other words, \(\phi'(q)\) is the contour line of \(\tilde{K}(u, q)\) corresponding to the level \(\lambda\).

In general, Equation (1.159) admits more than one real solution. Let us consider, for example, a function whose contour plot is reported in Figure 1A.1. We see that in the example reported in Figure 1A.1 a) there are two solutions...
for $x \in [x_0, x_1]$. In Figure 1A.1 b), we observe that there are two solutions for $x \in [x_0, x_2]$ and $x \in [x_3, x_1]$ and four solutions for $x \in [x_2, x_3]$.

If we reincorporate the eigenvalue index $i$ in our expressions, and we indicate with $p_{i,m}(q)$ one of the solutions of

$$
\tilde{K}(p_{i,m}(q), q) = \lambda_i,
$$

(1.162)

the first order expression for the eigenvalues of an inhomogeneous system assumes the form

$$
\psi_i(x) = \sum_m A_{i,m} e^{i\phi_{i,m}(x)},
$$

(1.163)

where $m$ runs over all the solutions, for the given $x$, and

$$
\phi_{i,m}(x) = \int_{x_0}^x p_{i,m}(y) dy + \phi_{i,m}(x_0).
$$

(1.164)

The initial phases $\phi_{i,m}(x_0)$ can be neglected without any loss of generality, as they may be incorporated in the amplitudes $A_{i,m}$. On their hand, the only constraint on the amplitudes $A_{i,m}$ is that the eigenfunctions have unit norm.

It is important to remark that not all values of $\lambda_i$ are eigenvalues of the system. In fact, if we denote by $A_i$ the area enclosed within the contour line associated to the $i$-th eigenvalue, the $i$-th eigenvalue must respect the following rule

$$
A_i = (2i + 1)\pi.
$$

(1.165)

1.7.0.6 Second order solution

Let us consider now the second order expansion (1.156) and see how to get a refined solution of (1.152). Substituting (1.156) and the expansion (1.153), up to the linear term in $\epsilon$, in (1.152) and setting $u = x - y$, we get the following integral

$$
\int_{-\infty}^{\infty} \left[ K_0(u, q) - \frac{\epsilon}{2} u K_01(u, q) \right] \left[ A_0(q) - \epsilon u A_0'(q) \right] e^{i\frac{\phi(q)}{2}} e^{-ju\phi'(q)}

\cdot \left[ 1 + j\epsilon u^2 \frac{\phi''(q)}{2} \right] du

\approx A_0(q) e^{i\frac{\phi(q)}{2}} \int_{-\infty}^{\infty} K_0(u, q) e^{-ju\phi'(q)} du + \epsilon e^{i\frac{\phi(q)}{2}} \int_{-\infty}^{\infty} e^{-ju\phi'(q)}

\cdot \left\{ K_0(u, q) \left[ j\phi''(q) \frac{u^2}{2} A_0(q) - u A_0'(q) \right] - \frac{u}{2} K_01(u, q) A_0(q) \right\} du.
$$

(1.166)
We may seek for a solution of the previous equation using the same phase found in the first order solution and finding the amplitude $A_0(q)$ that nulls the term proportional to $\epsilon$ in the previous equation. To this end, it is useful to introduce the following function

$$K_p(p, q) := \frac{\partial K(p, q)}{\partial p} = -j \int_{-\infty}^{\infty} uK(u, q) e^{-ju p} du,$$  \hspace{1cm} (1.167)$$

where we have used (1.160). Using (1.167), we may write

$$\int_{-\infty}^{\infty} K(u, q) u e^{-ju \phi'(q)} du = j K_p(\phi'(q), q). \hspace{1cm} (1.168)$$

Furthermore, deriving $K_p(\phi'(q), q)$ with respect to $q$, we obtain

$$j \frac{\partial K_p(\phi'(q), q)}{\partial q} = \int_{-\infty}^{\infty} uK(u, q) e^{-ju \phi'(q)} du$$
$$- \phi''(q) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^2 K(u, q) e^{-ju \phi'(q)} du. \hspace{1cm} (1.169)$$

Therefore, substituting (1.169) and (1.168) in the last line of (1.166), we can write the integrand function as

$$A_0(q) \tilde{K}_p(\phi'(q), q) + A_0(q) \frac{\partial K_p(\phi'(q), q)}{\partial q} \hspace{1cm} (1.170)$$

The function $A_0(q)$ that nulls the previous expression is a possible solution of our system, as it nulls the last line of (1.166). Equation (1.170) can be rewritten, equivalently, as

$$\tilde{K}_p^{1/2}(\phi'(q), q) \left\{ A_0(q) \tilde{K}_p^{1/2}(\phi'(q), q) + A_0(q) \frac{\partial K_p(\phi'(q), q)}{\partial q} \frac{1}{\tilde{K}_p^{1/2}(\phi'(q), q)} \right\}. \hspace{1cm} (1.171)$$

Assuming $\tilde{K}_p^{1/2}(\phi'(q), q) \neq 0$, $A_0(q)$ must be solution of the following differential equation

$$\frac{\partial}{\partial q} \left[ A_0(q) \tilde{K}_p^{1/2}(\phi'(q), q) \right] = 0. \hspace{1cm} (1.172)$$

The solution of this equation is

$$A_0(q) = \frac{1}{\tilde{K}_p^{1/2}(\phi'(q), q)} \hspace{1cm} (1.173)$$
Combining (1.173) with (1.155), we can state that the second order expression for the eigenfunctions of a weakly inhomogeneous system is then

$$
\psi_i(q; \epsilon) = \sum_m e^{j\phi_{i,m}(q)} \frac{1}{\sqrt{K_p(\phi_{i,m}(q), q)}},
$$

(1.174)

where the instantaneous phase $\phi_{i,m}(q)$ of each component is such that the corresponding derivative $\phi'_{i,m}(s)$ is a solution of (1.159).

The previous approximation breaks down for values of $q$ where $K_p(\phi'(q), q) = 0$. These points are turning points and are indicated by the dots in Figure 1A.1. Around such points, it is necessary to higher order developments.

**TIME-VARYING SYSTEMS**

We specialize now some of the results achieved in the previous section to time-varying systems. In particular, we wish to give a physical interpretation of the results shown above. For time-varying channels, the kernel is related to the channel impulse response through the following equation

$$
K(t, \theta) = \int_{-\infty}^{\infty} h(t, t - \tau) h^*(\theta, \theta - \tau) d\tau.
$$

(1.175)

Hence, proceeding as in (1.149), we can introduce the function $K(p, q)$ such that

$$
K(t, \theta) = K(t - \theta, \frac{\epsilon}{2}(t + \theta)),
$$

(1.176)

or, equivalently, setting $p = t - \theta$ and $q = t + \theta$,

$$
K(p, q) = K\left(\frac{p}{2} + \frac{q}{\epsilon}, -\frac{p}{2} + \frac{q}{\epsilon}\right).
$$

(1.177)

It is also useful to introduce the function

$$
K(u, q) := \int_{-\infty}^{\infty} K(p, q) e^{-j2\pi up} dp
$$

$$
= \int_{-\infty}^{\infty} K\left(\frac{p}{2} + \frac{q}{\epsilon}, -\frac{p}{2} + \frac{q}{\epsilon}\right) e^{-j2\pi up} dp
$$

$$
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h\left(\frac{p}{2} + \frac{q}{\epsilon}, \frac{q}{2} - \frac{p}{\epsilon}\right) h^*\left(-\frac{p}{2} + \frac{q}{\epsilon}, \frac{q}{2} - \frac{p}{\epsilon}\right) e^{-j2\pi up} dp dt.
$$
It is interesting to express $K(u, q)$ as a function of the channel transfer function $H(t, f)$. Substituting in the previous expression $h(t, \tau)$ with $\int_{-\infty}^{\infty} H(t, f)e^{j2\pi f \tau} df$, we get

$$
\hat{K}(f, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H \left( \frac{p}{2} + \frac{t}{\epsilon}, \nu \right) H^* \left( -\frac{p}{2} + \frac{t}{\epsilon}, \nu \right) e^{-j2\pi fp} dp d\nu. \quad (1.178)
$$

We wish to find out the equation of the line where $\hat{K}(f, t)$ is constant, as a function of $H(t, f)$. Setting $f = f(t) := \phi'(t)/2\pi$ and considering that, for $\epsilon \to 0$ we have $H \left( \frac{t}{2} + \frac{\nu}{\epsilon}, \nu \right) \approx H \left( \frac{t}{\epsilon}, \nu \right)$, we can rewrite $\hat{K}(f, t)$ as

$$
\hat{K}(f(t), t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H \left( \frac{t}{2} + \frac{t}{\epsilon}, \nu \right) H^* \left( \frac{t}{2} + \frac{t}{\epsilon}, \nu \right) e^{-j2\pi f(t)p} dp d\nu
$$

$$
\approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |H \left( \frac{t}{\epsilon}, \nu \right)|^2 \int_{-\infty}^{\infty} e^{-j2\pi f(t)p} dp d\nu = \lambda^2
$$

$$
\approx \int_{-\infty}^{\infty} |H \left( \frac{t}{\epsilon}, f(t) \right)|^2 \delta(f(t)) d\nu = |H \left( \frac{t}{\epsilon}, f(t) \right)|^2. \quad (1.179)
$$

Hence, condition becomes $\hat{K}(f, t) = \lambda^2$

$$
|H \left( \frac{t}{\epsilon}, f(t) \right)|^2 = \lambda^2. \quad (1.180)
$$

In words, the eigenfunctions of a slowly varying channel are composed of the superposition of components whose instantaneous frequency is given by the curve, in the time-frequency domain, where the modulus of the channel transfer function is constant and equal to the eigenvalue associated to the eigenfunction.

Proceeding similarly with the instantaneous envelope $a(t)$, we find out that

$$
a(t) = \left[ \frac{1}{\sqrt{\frac{\partial |H(t, f)|^2}{df}}}_{f=\phi'(t)/2\pi} \right]. \quad (1.181)
$$

APPENDIX 1A.2: TIME-FREQUENCY REPRESENTATIONS

We recall here a few basic properties of time frequency distributions (the interested reader may refer to [22] or [21], for example). Within the generalized Cohen’s
class of time-frequency distributions (TFD), the so called Wigner-Ville distribution (WVD) plays a prominent role because all other TFD’s can be derived from the WVD through a convolution in the time-frequency domain with the smoothing function characterizing the desired distribution. Given a signal \( s(t) \), its WVD is defined as

\[
W_s(t, f) = \int_{-\infty}^{\infty} s^*(t - \tau/2)s(t + \tau/2)e^{-j2\pi ft}\,d\tau.
\]  
(1.182)

The WVD satisfies the following properties.

1. **Moyal’s formula**: Given two signals \( x(t) \) and \( y(t) \), their scalar product is preserved in the time-frequency domain, in the sense that the following property, known as Moyal’s formula, holds true:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_x(t, f)W_y(t, f)\,dtdf = \left| \int_{-\infty}^{\infty} x^*(t)y(t)\,dt \right|^2.
\]  
(1.183)

2. **Inversion formula**: Given the WVD \( W_s(t, f) \) of a signal \( s(t) \), it is possible to recover \( s(t) \) from \( W_s(t, f) \), up to a scalar factor, using the following inversion formula:

\[
s(t) = \frac{1}{s^*(0)} \int_{-\infty}^{\infty} W_s(t, f)e^{j2\pi ft}\,df.
\]  
(1.184)

3. **Moments**: The instantaneous frequency \( f_s(t) \) of a complex signal \( s(t) \) is the center of gravity of its WVD along \( f \):

\[
f_s(t) = \frac{\int_{-\infty}^{\infty} fW_s(t, f)\,df}{\int_{-\infty}^{\infty} W_s(t, f)\,df}.
\]  
(1.185)

By duality, the group delay \( t_s(f) \) is the center of gravity of the WVD along the \( t \):

\[
t_s(f) = \frac{\int_{-\infty}^{\infty} tW_s(t, f)\,dt}{\int_{-\infty}^{\infty} W_s(t, f)\,dt}.
\]  
(1.186)
Channel models and modes