Decentralized Resource Allocation in Femtocell Networks Based on Game Theory and Markov Modeling

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Abstract

The potential massive deployment of femto access points requires femtocell networks to be able to self-configure, with a special attention to interference management. Within this context, game theory represents a solid tool to devise decentralized algorithms for dynamic resource allocation. In this paper, we propose alternative game-theoretic techniques that exploit the backhaul link among femto-access points to set up local coordination games which provide performance advantages with respect to purely competitive games. Our optimization techniques incorporate a Markovian model of the interference activity and propose alternative pricing mechanisms to allocate power in the joint time-frequency plane.

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1 Introduction

Femtocell networks are composed of cells having a coverage radius in the order of tens of meters, which provide enhanced indoor coverage through the use of femto-access points (FAP) or Home Base Stations (HBS) [1], [2]. These stations are low power base stations connected to the cellular operator’s network via a digital subscriber line (DSL) or cable modem. The femto mobile user communicates with a FAP via a wireless link, while the FAP forwards traffic to the macro base stations (MBS) through a wired line. A typical scenario is sketched in Fig.1. The wired link among FAPs is the backhaul link and can be exploited for setting up a local coordination among FAPs and to exchange information with the MBS.

Femtocells are becoming more and more attractive as they offer benefits for both operators and subscribers. The operators see femtocells as a way to off-load traffic from the macro cellular network to the wired lines, thus reducing the cost for the wireless infrastructure, and to improve indoor coverage without deploying expensive macro base stations. On the other hand, subscribers see femtocells as a way to get higher quality services, either higher data throughput or better voice quality, thanks to the capillary indoor coverage of femto-access points, which avoid the wall penetration losses.

Differently from Wi-Fi, femtocell networks are fully compliant with cellular standards, so that the femto user keeps using his cellular handset while being totally unaware whether his communication is routed through a femto-access point or goes directly to a macro base station. But, differently from macro base stations, FAPs are typically installed by the subscribers and maintained without global planning, with no special consideration about traffic demands or interference with other cells, either femto or macro cells. Hence, a potential massive deployment of FAPs might induce an intolerable interference from femto to macro users, as well as from femto to femto users. Interference management is then arguably the major challenge to be faced by femtocell networks.

In principle, the optimal solution to interference management would require an accurate
global planning. However, a centralized planning would require the exchange of a huge amount of data among the many femto-access points and the macro base stations. This would induce an excessive signaling traffic. It is then of special interest to devise decentralized mechanisms able to adapt resource allocation dynamically in order to limit interference adequately and get the advantages offered by the capillary deployment of FAPs.

A fundamental tool to devise innovative decentralized resource allocation strategies is game theory, a branch of mathematics studying interactive decision problems connected to multi-objective optimization. Game theoretic approaches have been proposed for the multicarrier interference channel [3, 4, 5] and, lately, for cognitive radios [6]. One of topics that received more attention is the existence and uniqueness of equilibrium points of the game characterizing the interaction among the FAPs [3]. A possible form of equilibrium is the celebrated Nash Equilibrium (NE), indicating the condition in which every player has no incentive to unilaterally deviate from his strategy, given the strategies of the other players. However, a NE, just because of its purely competitive nature, could be Pareto-inefficient. It is then of interest to check if there are strategies to modify the game in order to move the NE’s of the modified game towards the Pareto optimal boundary. One of the mechanisms to be used to achieve such a goal is pricing, which requires some exchange of information among players. Differently from cognitive radios, where this exchange of information is not possible, a possible local coordination among FAPs is made possible in femtocell networks through the backhaul link which creates an underlying wired network connecting FAPs and MBSs, as proposed in [7]. Aimed at allocating power optimally in the joint time-frequency domain, following the current trend in 3G systems and their evolution like WiMax and LTE, it is of particular interest to have a model for the interference activity in the time-frequency plane. As suggested in previous works on cognitive radios, a useful statistical model for the interference activity is the Markov model, see, e.g. [8] and its references. The main contributions of this paper are the following. We start with a static model in Section 2.1 to propose a decentralized spatial reuse mechanism, based on
iterative water-filling. Then, we propose in Section 2.2 pricing mechanisms for minimum power games, exploiting the backhaul link for local coordination among FAPs. In Section 3, we propose maximum rate and minimum power statistical games incorporating Markov interference activity models. Finally, we generalize the pricing mechanisms to these games to improve performance with respect to purely competitive games.

2 Decentralized resource allocation through game theory

In this section, we recall the maximum rate game, leading to the well known iterative water filling algorithm, and show how to use it to achieve spatial reuse of frequency. Then, we recall the minimum power game and introduce pricing mechanisms to improve its performance.

2.1 Max-rate game

To study the interference mechanism, it is useful to introduce what we call the interference graph, defined as the graph whose vertices are FAPs and where there is an edge between two vertices only if the relative FAPs are within the coverage radius of each other. Given the limited transmit power of each FAP, the interference graph is typically a sparse graph (i.e., with a number of links much smaller than the maximum number of possible links). It is worth emphasizing that, given the underlying OFDMA framework, two neighbor nodes in the interference graph do not necessarily interfere with each other. They could in fact allocate power over orthogonal channels and thus avoid to interfere. What the interference graph specifies is the pairs of nodes who do not interfere with each other, irrespective of their power allocation, because they are out of reach of each other: These are the nodes who are not neighbors in the interference graph. For each node $q$, we denote by $\mathcal{N}_q$ the set of nodes having node $q$ within their coverage area. We denote by $\mathbf{p}_q = (p_{q1}, \ldots, p_{qN})$ the power vector of user $q$, whose element $p_{qk}$ is the power transmitted by node $q$ over the $k$th subcarrier. Adopting symbols common in game theory, $\mathbf{p}_{-q}$ denotes the vector containing the power vectors of all users except user $q$. 

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The rate of user $q$ can then be written as

$$R_q(p_q, p_{-q}) = \sum_{k=1}^{N} \log \left( 1 + \frac{1}{\Gamma_q} \frac{|H_k^{qq}|^2 p_k^q}{\sigma_{qk}^2 + \sum_{r \in N_q} |H_k^{rq}|^2 p_k^r} \right)$$

(1)

where $H_k^{rq}$ is the channel transfer function of the $k$th subchannel between the $r$-th transmitter and the $q$-th receiver; $\sigma_{qk}^2$ is the variance (power) received on the $k$-th subchannel including receiver noise and power coming from the macro users; $\Gamma_q$ is the SNR margin, dictated by the requirement on the bit error rate (BER).

One of the game formulations that have received more attention in the wireless communication literature consists in the maximization, for each user, of the rate $R_q$, as given in (1), under a power constraint [9]. Denoting with $P_q$ the maximum power of user $q$, the feasible set for user $q$, i.e. the set of powers satisfying the power limit constraint is $\mathcal{P}_q \equiv \left\{ p_q \in \mathbb{R}^N : \sum_{k=1}^{N} p_k^q \leq P_q, 0 \leq p_k^q \leq p_{q,\text{max}}^q(k), \forall k = 1, \ldots, N \right\}$, where $p_{q,\text{max}}^q(k)$ denotes a power mask constraint that limits the maximum transmit power over each channel. The maximum rate game is the game

$$\mathcal{G}_1 = \{ \Omega, \{ \mathcal{P}_q \}_{q \in \Omega}, \{ R_q(p_q, p_{-q}) \}_{q \in \Omega} \}$$

(2)

where $\Omega \triangleq \{1, \ldots, Q\}$ is the set of $Q$ players, $\{ \mathcal{P}_q \}_{q \in \Omega}$ is the set of admissible strategies (i.e., feasible power vectors) and $\{ R_q(p_q, p_{-q}) \}_{q \in \Omega}$ is the set of utilities. The goal of each player in this game is to choose the power vector that solves the following constrained optimization problem

$$\max_{p_q} \quad R_q(p_q, p_{-q})$$

subject to \( \sum_{k=1}^{N} p_k^q \leq P_q, 0 \leq p_k^q \leq p_{q,\text{max}}^q(k), \forall k = 1, \ldots, N \)

given the strategies adopted by the other players. Game $\mathcal{G}_1$ has been thoroughly studied and one of the topics that received more attention is the existence of the so called Nash Equilibrium, indicating the condition under which every player has no incentive to unilaterally deviate from his strategy, given the strategies of the other players. In formulas, a NE for game $\mathcal{G}_1$ is defined as the power vector $p^* := (p_q^*, p_{-q}^*)$ such that

$$R_q(p_q^*, p_{-q}^*) \geq R_q(p_q, p_{-q}^*), \forall p_q \in \mathcal{P}_q, \forall q \in \Omega.$$
The conditions for the existence and uniqueness of the NE of game $G_1$ have been thoroughly studied \cite{9}, \cite{3}. One possible iterative algorithm converging to a NE is the popular *iterative water-filling* (IWF) algorithm \cite{9}, \cite{3}. In the high-interference regime, IWF tends to orthogonalize the channels associated to different links, in order to reduce as much as possible the interference, even if reducing the number of channels. We exploit this orthogonalizing capability of IWF to devise a decentralized resource allocation algorithm able to provide spatial reuse of frequency. This can be achieved by “fictitiously” increasing the parameter $\Gamma_q$, which is equivalent to increase the weight of interference in the rate computation. As $\Gamma_q$ increases, IWF tends to make all channels orthogonal, in order to null the interference. In general, if the interference graph is fully connected, making all channels orthogonal requires a number of carriers at least equal to the number of potential interferers. However, the interference graph is typically very sparse, just because of the low transmit power and of the attenuation losses due to wall penetration. As a consequence, to avoid conflicts it is necessary to have a number of subcarriers greater than the maximum degree of the interference graph. This sentence is corroborated by the plots reported in Fig. 2, showing the ratio between the number $N_{\text{conf}}$ of links in conflict, after running IWF with a deliberately high value of $\Gamma_q$, and the number $N_{\text{tot}}$ of total links vs. the number of subcarriers $N$. The three plots refer to topologies composed of 10, 50 and 100 FAPs. From Fig. 2 we can check that, as the number of subcarriers increases, IWF is able to assign orthogonal channels to all the FAPs, as the number of conflicts tends to zero. Interestingly, if we assume a simple statistical model for the network topology, it is also possible to predict the number of channels necessary to avoid conflicts theoretically, with high probability, without knowing the position of the FAPs. A simple, yet meaningful, statistical model for femtocell networks is the random geometric graph, where the vertices (FAPs) are located randomly and there is a link between two vertices if their distance is less than a prescribed coverage radius $r_0(n)$, which depends on the number of nodes $n$. If the nodes are uniformly located over a unit square and the radius follows the law $r_0(n) = \sqrt{(\log n + c(n))/\pi n}$, then, asymptotically, if $c(n)$ tends to
infinity as \( n \) goes to infinity, the network is asymptotically connected with high probability, i.e. with probability going to one, as \( n \) tends to infinity [10]. For example, taking \( c(n) = \log n \), one can find the coverage radius that guarantees connectivity with high probability. In such a case, the average number of neighbors is the same for every node (neglecting border effects) and equal to \( |\mathcal{N}_i| = 2 \log n \). This means that, if we take a number of carriers \( N > 2 \log n + 1 \), it is possible to have no conflicts among neighbors. To check this property, in Fig. 2, we reported three dots, one for each number of nodes, marking the values \( 2 \log n + 1 \) on the abscissa axis. We can verify from Fig. 2 that, indeed, for \( N \) slightly greater than \( 2 \log n + 1 \), the probability of having a conflict tends to zero. This means that, for a random geometric graph, it is possible to achieve a large reuse factor, in the order of \( n/ \log n \), using a purely decentralized algorithm and without knowing the nodes positions.

Reaching a NE for game \( \mathcal{G}_1 \) can be obtained through a totally decentralized way. However, game \( \mathcal{G}_1 \) may possess multiple equilibria and they may not be Pareto-efficient\(^1\), because of the purely competitive nature of game \( \mathcal{G}_1 \). To improve upon the performance of the NE of game \( \mathcal{G}_1 \), it has been proposed to modify the utility function of each user in order to induce the players to take care of a social utility function, rather than being purely selfish. For example, in [11, 12] it has been proposed to modify the utility function of each player so as to maximize the sum of all users’ rates. In principle, this change should require a centralized solution. Nevertheless, the authors of [11] showed that the solution of the sum-rate game can be still achieved in decentralized form, provided that the players exchange some parameters, the so called prices. These parameters induce a penalty, on each player utility proportional to the rate decrease that the player strategy induces on the other players. Introducing pricing mechanisms in femtocell networks is well suited, as the FAPs can exchange information through the backhaul link. 

\(^1\)We recall that a set of strategies is \textit{Pareto efficient}, or \textit{Pareto optimal}, if it is no possible to make at least some player better off without making any other player worse off. Given the whole set of feasible strategies, i.e. the strategies satisfying the system constraints, the \textit{Pareto boundary} is defined as the set of choices that are Pareto efficient. If an equilibrium point belongs to the Pareto boundary, the equilibrium is said to be efficient.
will show next that, thanks to the sparse structure of the interference graph, every FAP has to exchange pricing coefficients only with its neighbors.

2.2 Min-power game

Given the need to limit the power emitted by FAPs, an alternative approach to rate maximization consists in minimizing the transmit power necessary to fulfill the rate requirement of each user, as proposed for example in [13] in Gaussian parallel interference channels. In this case, denoting by $R^0_q$ the rate required by user $q$, the set of feasible strategies of user $q$ is

$$
\mathcal{P}_q(p_{-q}) = \{ p_q \in \mathbb{R}^N : R_q(p_q, p_{-q}) \geq R^0_q, \ 0 \leq p^q_k \leq p^\text{max}_q(k), \ k = 1, \ldots, N \}. \tag{5}
$$

The utility of each user is the transmit power, i.e. $u_q(p_q) = \sum_{k=1}^{N} p^q_k$. Hence, the min-power game is

$$
\mathcal{G}_2 = \{ \Omega, \{ \mathcal{P}_q(p_{-q}) \}_{q \in \Omega}, \{ u_q(p_q) \}_{q \in \Omega} \}. \tag{6}
$$

Each player chooses the strategy that solves the following constrained problem

$$
\begin{align*}
(P_2) \quad \min_{p_q} & \quad u_q(p_q) \\
\text{subject to} & \quad R_q(p) \geq R^0_q; \ 0 \leq p^q_k \leq p^\text{max}_q(k), \ k = 1, \ldots, N.
\end{align*} \tag{7}
$$

It is worth to point out that the feasible set of every player now depends on the strategies chosen by the other players. In other words, while the max-rate game has coupled utility functions and uncoupled constraints, the min-power game has uncoupled utilities and coupled constraints. This makes the problem of finding an NE for the min-power game harder to solve. In this case, the possible equilibrium points of the game are called Generalized NE (GNE), to point out the coupled nature of the constraints. The min-power game has been studied in [13]. Again, as with game $\mathcal{G}_1$, the GNE’s of game $\mathcal{G}_2$ may be Pareto-inefficient, because of its purely competitive nature. Hence, it is worth asking whether it is possible to modify game $\mathcal{G}_2$ in order to improve its performance. This case is different from game $\mathcal{G}_1$ because, even if unknowingly, every player of game $\mathcal{G}_2$ is already pursuing a social utility goal. In fact, game $\mathcal{G}_2$ is a generalized exact
potential game [14]. We recall that a game with utility function \( u_q(p_q, p_{-q}) \) is an exact potential game if there exists a function \( U(p_q, p_{-q}) \), called the potential, such that for all \( q \in \Omega \)

\[
u_q(x_q, p_{-q}) - u_q(y_q, p_{-q}) = U(x_q, p_{-q}) - U(y_q, p_{-q}), \forall x_q, y_q \in P(p_{-q}).
\]

It is easy to check that the potential of game \( G_2 \) is simply the sum of all the powers: \( U(p) = \sum_{q=1}^{Q} \sum_{k=1}^{N} p_{q}^{k} \). More specifically, since the constraints of \( G_2 \) are coupled, \( G_2 \) is a generalized potential game (GPG) [15]. Hence, since in this case each player is already pursuing a social goal (minimization of the total radiated power), we may wonder whether it is still possible to improve the performance of game \( G_2 \) by incorporating some pricing mechanism, similarly to what done for game \( G_1 \) in [12]. To this end, we reformulate game \( G_2 \), as follows

\[
\begin{align*}
\min_{p} & \quad \sum_{q=1}^{Q} u_q(p_q) \\
\text{(PC)} & \quad \text{subject to } R_q(p) \geq R_q^0, \quad \forall q \in \Omega \\
& \quad 0 \leq p_q^{k} \leq p_{q}^{\max}(k), \quad k = 1, \ldots, N, \quad \forall q \in \Omega
\end{align*}
\]

In principle, the solution of this problem requires the existence of a central station that has all the necessary information. Nevertheless, a limited exchange of information among nearby FAPs is sufficient to implement a decentralized solution of \( (PC) \), which requires only local coordination among nearby FAPs. For the optimization problem \( (PC) \) to be meaningful, it is necessary to check first that the feasible set is nonempty. In Appendix A we give sufficient conditions guaranteeing that the feasible set of \( (PC) \) is nonempty and compact so that the problem admits at least a solution point. It can be proved that any local optimum of \( (PC) \) is a regular point\(^2\) then it must satisfy the Karush-Kuhn Tucker (KKT) necessary conditions [16]. In particular, the Lagrangian associated to problem \( (9) \) is

\[
\mathcal{L}(p, \lambda, \mu, \nu) = \sum_{q=1}^{Q} \sum_{k=1}^{N} p_{k}^{q} - \sum_{q=1}^{Q} \lambda_q(R_q(p) - R_q^0) - \sum_{q=1}^{Q} \sum_{k=1}^{N} \mu_{k}^{q} p_{k}^{q} + \sum_{q=1}^{Q} \sum_{k=1}^{N} \nu_{k}^{q}(p_{k}^{q} - p_{q}^{\max}(k))
\]

\(^2\)A feasible point is said to be regular if the equality constraints gradients and the active inequality gradients are linearly independent [16].
This problem can be solved locally, by FAP \( \lambda \) the local Lagrangian coefficient \( \tilde{r} \)

\[
\frac{\partial \mathcal{L}(p_\lambda \mu \nu)}{\partial p_k} = 1 - \lambda_q \frac{\partial R_q}{\partial p_k^q} - \sum_{r \neq q} \lambda_r \frac{\partial R_r}{\partial p_k^r} - \mu_k^q + \nu_k^q = 0
\]

\[
0 \leq \mu_k^q \perp p_k^q \geq 0
\]

\[
0 \leq \nu_k^q \perp p_k^{\max}(k) - p_k^q \geq 0
\]

\[
0 \leq \lambda_q \perp R_q(p) - R_q^0 \geq 0
\]

with

\[
\frac{\partial R_q}{\partial p_k^q} = \frac{|H_k^q|^2}{\sigma_{q k}^2 + I_k^q + |H_k^q|^2 p_k^q}; \quad \frac{\partial R_r}{\partial p_k^q} = -\frac{|H_k^r|^2 |H_k^q|^2 p_k^q}{(\sigma_{r k}^2 + I_k^r)(\sigma_{r k}^2 + I_k^r + |H_k^r|^2 p_k^r)} \mathbb{I}(r \in \mathcal{N}_q)
\]

where \( I_k^q = \sum_{i \in \mathcal{N}_q} |H_k^i|^2 p_k^i \) and \( \mathbb{I}(X) \) is a logical function equal to one, if \( X \) is true, or zero, otherwise. Proceeding as in [11], it is useful to introduce the price coefficients:

\[
\pi_k^r := -\frac{\partial R_r(p)}{\partial I_k^r},
\]

which is proportional to the marginal decrease of user \( r \)'s rate because of an increase of the \( q \)th node’s transmit power, as \( \frac{\partial R_r(p)}{\partial p_k^q} = -\pi_k^r \frac{\partial I_k^r}{\partial p_k^q} = -\pi_k^r |H_k^q|^2 \). If the prices are assumed to be constant with respect to \( p_k^q \), solving problem (11) with respect to \( p_q \) is equivalent to solving the following local problem

\[
\text{(P)} \quad \min_p \quad \sum_{k=1}^N (1 + \sum_{r \in \mathcal{N}_q} \lambda_r \pi_k^r |H_k^q|^2) p_k^q
\]

subject to \( R_q(p) \geq R_q^0, 0 \leq p_k^q \leq p_k^{\max}(k), \quad k = 1, \ldots, N. \)

This problem can be solved locally, by FAP \( q \), provided that its neighbors send the coefficients \( \lambda_r \pi_k^r |H_k^q|^2 \). The local solution, given the powers used by all other FAPs, is

\[
p_k^q = \left[ \frac{\lambda_q}{1 + b_k^q} \right] \left[ \frac{\sum_{r \in \mathcal{N}_q} \lambda_r \pi_k^r |H_k^q|^2}{\sigma_{q k}^2 + I_k^q + |H_k^q|^2} \right] p_k^{\max}(k)
\]

where \([x]_a^b\) denotes the projection of \( x \) into the interval \([0, p_k^{\max}(k)]\) and \( b_k^q = \sum_{r \in \mathcal{N}_q} \lambda_r \pi_k^r |H_k^q|^2 \).

The local Lagrangian coefficient \( \lambda_q \) must satisfy the equality \( \lambda_q (R_q(p) - R_q^0) = 0 \), with \( R_q(p) \geq \frac{\partial R_q}{\partial p_k^q} = 1 - \lambda_q \frac{\partial R_q}{\partial p_k^q} - \sum_{r \neq q} \lambda_r \frac{\partial R_r}{\partial p_k^r} - \mu_k^q + \nu_k^q = 0 \)

\( a \perp b \) means that the vectors \( a \) and \( b \) are orthogonal.

\( ^4 \) In general, the assumption of \( \pi_k^r \) to be constant with respect to \( p_k^q \) is only an approximation. Nevertheless, the resulting algorithm provides significant performance improvement with respect to the purely competitive game.
Algorithm 1:

S.0: Choose any feasible power allocation $p^0 = (p^0_q, \ldots, p^0_Q)$ and set $n = 0$;

S.1: If $p_q(n)$ satisfies a suitable termination criterion then STOP, otherwise

S.2: Set $n := n + 1$ and for $q = 1, \ldots, Q$ compute $p^q_k(n)$ from (15), using (16);

S.3 Compute $\lambda_q$ and $\pi^q_k$ and broadcast $\lambda_q\pi^q_k|H^q_{ik}|^2$ to its neighbors with index $i \in \mathcal{N}_q$;

S.4 Set $p(n) = (p_1(n), \ldots, p_Q(n))$ and go to S.1.

Table 1: Min-power algorithm with pricing

$R^0_q$. From (15) $\lambda_q$ must be strictly greater than zero, otherwise the powers $p^q_k$ will all be equal to zero and this would contradict the inequality $R_q(p) \geq R^0_q$. Hence, $R_q(p) = R^0_q$ and, as a consequence, $\lambda_q$ can be found as the coefficient that guarantees $R_q(p) = R^0_q$. Some power coefficients $p^q_k$ can be null. Let us denote by $\mathcal{D}_q$ the set of subcarriers where user $q$ allocates a nonnull power $p^q_k < p^\text{max}_q(k)$ and by $\mathcal{D}_q'$ the set of subcarriers where user $q$ allocates a power $p^q_k = p^\text{max}_q(k)$. After a few algebraic manipulations, we can express $\lambda_q$ in closed form as

$$
\lambda_q = e^{\frac{1}{|\mathcal{D}_q|} \left[ R^0_q - \sum_{k \in \mathcal{D}_q} \log \left( \frac{|H^q_{ik}|^2}{(1 + p^\text{max}_q(k)|H^q_{ik}|^2)} \right) - \sum_{k \in \mathcal{D}_q'} \log \left( 1 + p^\text{max}_q(k)|H^q_{ik}|^2 \right) \right]}.
$$

(16)

In summary, the decentralized min-power algorithm is described in Table 1. An example of performance improvement achievable with the pricing mechanism for the min-power game is shown in Figs. 3 and 4, which report the sum of the radiated power and the sum rate, respectively, obtained with and without pricing, vs. the number of iterations. The results refer to a scenario with three interfering FAPs. The results have been averaged over 200 independent Rayleigh channel realizations, with a number of carriers for each transmitting FAP equal to 72 and a channel length of 4 taps. From these figures, it is evident the advantage of using pricing mechanisms to reduce the radiated power, still maintaining the same service quality.
3 Dynamic resource allocation under Markovian interference model

Femtocells must be fully compliant with cellular standards. Given the current evolution of 3G systems (see, e.g. WiMax and LTE), it is of interest to look at power allocation techniques in a time-frequency frame. Typically, channel and interference power are sensed at the beginning of each frame and assumed constant over the frame. However, while the constant channel assumption is perfectly reasonable given the low indoor mobility, the interference from macro-users may vary along the frame. As a consequence, the correct power allocation across time and frequency would require a non-causal knowledge of the interference, which of course is not available. To circumvent this inconvenient, we propose a time-frequency resource allocation based on a Markov modeling of the interference activity on each frequency subchannel. More specifically, we assume that the activity of the macro users over each subchannel is modeled as two-state homogeneous Discrete Time Markov Chains (DTMC), with known (estimated) transition probabilities and that the activities over different channels are statistically independent of each other. Each FAP measures the interference power from the macro network over each subchannel in the first time slot of each frame. We use the binary variable $S_{k,m}$ to indicate the activity over the $k$th subchannel, at time $m$: $S_{k,m} = 1$ if the subchannel is busy, with interference power $\sigma^2_{I_q}(k,m)$, while $S_{k,m} = 0$ if the subchannel is idle. In the following, we generalize the max-rate and min-power games of Section 2 to the case of Markov interference activity.

3.1 Maximum expected rate game

Since the interference level along the frame is unknown, we propose to modify game $\mathcal{G}_1$ by substituting the utility function with the expected value of the rate $\bar{R}_q(p_q,p_{-q})$, conditioned to the observation performed on the first time slot. Denoting with $\mathbb{P}\{S_{k,m} = 0\} = \beta_{k,m}$ and $\mathbb{P}\{S_{k,m} = 1\} = \gamma_{k,m}$ the probabilities that channel $k$ is idle or busy, at time $k$, the expected rate
\[ \tilde{R}_q(p_q, p_{-q}) = \sum_{m=1}^{M} \sum_{k=1}^{N} \left[ \beta_{k,m} \log \left( 1 + p_{k,m}^q a_q^q(k,m) \right) + \gamma_{k,m} \log \left( 1 + p_{k,m}^q a_q^q(k,m) \right) \right], \quad (17) \]

with

\[ a_q^q(k,m) = \frac{|H_{k}^{qq}|^2}{\sigma_{n,q}^2(k) + \sum_{r \in N_{q}} p_{k,m}^r |H_{k}^{rq}|^2}, \quad \sigma_{n,q}^2(k) = \sigma_n^2 + \sum_{m \in M} p_{k,m}^q |H_{k}^{rq}|^2 + \sigma_r^2(k,m). \quad (18) \]

The probabilities \( \beta_{k,m} \) and \( \gamma_{k,m} \) evolve in time according to the following model

\[
\begin{pmatrix}
\beta_{k,m+1} \\
\gamma_{k,m+1}
\end{pmatrix} =
\begin{pmatrix}
\omega_k & 1 - \mu_k \\
1 - \omega_k & \mu_k
\end{pmatrix}
\begin{pmatrix}
\beta_{k,m} \\
\gamma_{k,m}
\end{pmatrix}, \quad m = 1 \ldots M - 1, k = 1 \ldots N, \quad (19)
\]

where \( \omega_k \) and \( \mu_k \) denote the idle-to-idle and busy-to-busy transition rates, respectively; \( \beta_{k,0} = 0 \) and \( \gamma_{k,0} = 1 \), if channel \( k \) is busy at time 0, or \( \beta_{k,0} = 1 \) and \( \gamma_{k,0} = 0 \), if channel \( k \) is idle at time 0. In (17), we distinguish between the interference from the macro-users, which is assumed as given, and the interference from other FAPs, which are modeled as competitive players. In the modified game, each player must solve the following local problem

\[
\hat{P}_1 : \quad \max_{p_q} \quad \tilde{R}_q(p_q, p_{-q}) \quad \forall q \in \Omega \quad \text{subject to} \quad p_q \in \tilde{P}_q \quad (20)
\]

where the feasible set of user \( q \) is

\[
\tilde{P}_q = \left\{ p_q \in \mathbb{R}^{NM \times 1} : \sum_{m=1}^{M} \sum_{k=1}^{N} p_{k,m}^q \leq P_q, 0 \leq p_{k,m}^q \leq p_{k,m}^{\max}(k), \forall k \in \{1, \ldots, N\}, m \in \{1, \ldots, M\} \right\}. \quad (21)
\]

Since the objective function in (20) is strictly concave in \( p_q \in \tilde{P}_q \), for any given \( p_{-q} \), and the feasible set \( \tilde{P}_q \) is compact and convex, game \( \tilde{G}_1 \) admits a non-empty solution set for any set of channels and transmit power constraints of the users. In [17] we reformulated this game as a Variational Inequality (VI) [18] and we applied the Iterative Gradient Projection Algorithm (IGPA) to solve it, deriving sufficient conditions for its convergence. Here, we show how to introduce a pricing mechanism to this new game. Proceeding as in Section 2.1, the modified
game, in this case, is
\[
\max_{\mathbf{p}_q} \tilde{R}_q(\mathbf{p}_q; \mathbf{p}_{-q}) - \sum_{m=1}^{M} \sum_{k=1}^{N} p^q_{k,m} \left( \sum_{r \in \mathcal{N}_q} \pi^r_{k,m} |H^r_k|^2 \right)
\]
\[
\text{s.t.} \quad \mathbf{p}_q \in \mathcal{P}_q.
\]
where the prices are now defined as
\[
\pi^r_{k,m} := -\frac{\partial \tilde{R}_r(\mathbf{p})}{\partial I^r_{k,m}(\mathbf{p}_{-r})}
\]
with $I^r_{k,m}(\mathbf{p}_{-r}) := \sum_{i \in \mathcal{N}_r} p^i_{k,m} |H^i_{k,m}|^2$. The best response of each FAP leads to power coefficients $p^q_{k,m}$ that, within the interval $[0, p^\text{max}_q(k)]$, must satisfy the following equation
\[
\tilde{a}^q(k, m)(p^q_{k,m})^2 + \tilde{b}^q(k, m)p_{k,m} + \tilde{c}^q(k, m) = 0
\]
where, denoting with $\nu_q$ the Lagrangian multiplier, we have set
\[
\tilde{a}^q(k, m) = (\nu_q + \sum_{r \in \mathcal{N}_q} \pi^r_{k,m} |H^r_k|^2)a^q_n(k, m)a^q_f(k, m)
\]
\[
\tilde{b}^q(k, m) = (\nu_q + \sum_{r \in \mathcal{N}_q} \pi^r_{k,m} |H^r_k|^2)(a^q_n(k, m) + a^q_f(k, m)) - a^q_n(k, m)a^q_f(k, m)
\]
\[
\tilde{c}^q(k, m) = \nu_q + \sum_{r \in \mathcal{N}_q} \pi^r_{k,m} |H^r_k|^2 - (a^q_n(k, m)\beta_{k,m} + a^q_f(k, m)\gamma_{k,m})
\]
We can verify that, $\forall \nu_q > 0$, we have $\tilde{b}^q(k, m)^2 - 4\tilde{a}^q(k, m)\tilde{c}^q(k, m) \geq 0$, and the only solution is
\[
p^q_{k,m} = \frac{-\tilde{b}^q(k, m) + \sqrt{\tilde{b}^q(k, m)^2 - 4\tilde{a}^q(k, m)\tilde{c}^q(k, m)}}{2\tilde{a}^q(k, m)}.
\]
More specifically, we get
\[
p^q_{k,m} = \begin{cases} 
0 & \text{if } \nu_q + \sum_{r \in \mathcal{N}_q} \pi^r_{k,m} |H^r_k|^2 \geq a^q_n(k, m)\beta_{k,m} + a^q_f(k, m)\gamma_{k,m} \\
\hat{p}^b_{k,m} & \text{if } \nu_q + \sum_{r \in \mathcal{N}_q} \pi^r_{k,m} |H^r_k|^2 < a^q_n(k, m)\beta_{k,m} + a^q_f(k, m)\gamma_{k,m}
\end{cases}
\]
and the optimal power allocation vector is $p^q_{k,m} = [\hat{p}^b_{k,m}]^{p^\text{max}_q(k)}$ where the multiplier $\nu_q$ is chosen in order to satisfy the constraint $\sum_{k=1}^{M} \sum_{m=1}^{N} [\hat{p}^b_{k,m}]^{p^\text{max}_q(k)} = P_q$.

The previous solution assumes, for each player, that the powers used by the other players are given. In practice, the game evolves with each FAP reacting to the choices of the other FAPs. It is then fundamental to prove the convergence of this iterative mechanism. In the following, we
present a version of the so called Modified Asynchronous Distributed Pricing algorithm (MADP) proposed in [12], adapted to our formulation.

To find the user’s best response, it is useful to rewrite (22) introducing a unique index $h$ so that the entries of the power vector $p_q$ are $p_q^h$ for $h = 1, \ldots, NM$. Then, defining the quantities

$$SNIR_{h}^{\beta_q} := \frac{p_q^h |H_{h}^{\beta_q}|^2}{\sigma_{n,q}^2(h) + \sum_{r \in N_q} p_r^h |H_{h}^{\beta_q}|^2}, \quad SNIR_{h}^{\gamma_q} := \frac{p_q^h |H_{h}^{\gamma_q}|^2}{\sigma_{n,q}^2(h) + \sum_{r \in N_q} p_r^h |H_{h}^{\gamma_q}|^2 + \sigma_{I_q}^2(h)}$$

we can derive the $q$-th user best response as

$$p_q^* = \frac{2c_q}{\sum_{r \in N_q} \pi_{r}^{\nu_q} |H_{h}^{\nu_q}|^2 + \nu_q - \eta_q^q} - p_q^h \quad \forall h = 1, \ldots, MN,$$  

where $c_q^h = \frac{\beta_q^{\gamma_q}SNIR_{h}^{\beta_q}}{1 + SNIR_{h}^{\beta_q}} + \frac{\gamma_q^{\beta_q}SNIR_{h}^{\beta_q}}{1 + SNIR_{h}^{\beta_q}}$ and $\nu_q$ and $\eta_q^q$ are the Lagrangian multipliers. Given this setting, the modified MADP algorithm is illustrated in Table 2.

Following similar arguments as [12], we proved in Appendix B that there exists a small enough step size values $\alpha_q(n)$ for which the MADP algorithm converges monotonically to a fixed point. The following example shows the effects of pricing on the maximum expected rate game formulated. In Figs. 5 and 6 we report the sum rate of two FAPs, in the case where the macro user activity is modeled as a two-state first order Markov chain, vs. the number of time slots. In particular, Fig. 5 refers to the purely competitive maximum expected rate game of Section 3.1, while Fig. 6 refers to the modified game including pricing. The results are obtained

<table>
<thead>
<tr>
<th>Algorithm 2: Modified Asynchronous Distributed Pricing</th>
</tr>
</thead>
<tbody>
<tr>
<td>S.0: Each user $q$ chooses an initial power profile in the set $\tilde{P}_q$ and set $n = 0$;</td>
</tr>
<tr>
<td>S.1: Each user computes its interference prices $\pi_q^h(n)$ for $h = 1, ..., MN$ and sends them to the other users;</td>
</tr>
<tr>
<td>S.2: At each time $n$, one user is randomly selected to maximize its utility function $\bar{R}<em>q$ and update its power profile given the other user’s power profiles $p</em>{-q}$ and price vectors $\pi_{r}^{\nu_q}(n)$ according to $p_q^h(n + 1) = p_q^h(n) + \alpha_q(n) (p_q^r - p_q^h(n))$ for $h = 1, ..., MN$, where $p_q^* n$ is given by (29);</td>
</tr>
<tr>
<td>S.3: Set $n = n + 1$, go to step S.1 and repeat until convergence is reached.</td>
</tr>
</tbody>
</table>

| Table 2: MADP Algorithm. |


by averaging over 100 independent Markov chains with idle-to-idle transition probabilities $\omega_k = \omega = 0.1$ and busy-to-busy probability $\mu_k = \mu = 0.2$. The number of subcarriers is set to 48, which is a multiple of the LTE Primary Resource Block (PRB). The three different curves in each figure indicate the sum rate obtained by assuming perfect (non-causal) knowledge of the macro-user activity, no knowledge at all, or statistical knowledge, i.e. knowledge of the Markov parameters. Both figures show that the statistical knowledge (estimation) of the interference activity parameters (Markov transition rates) yields a performance advantage over the case with no information and brings the performance close to the ideal case of perfect non-causal knowledge of the interference activity. Furthermore, comparing Figs. 5 and 6, it is evident the gain achieved with the introduction of pricing.

3.2 Min-power game subject to Markovian interference

Let us consider now the generalization of the min-power game of Section 2.2 to the Markovian interference case. The utility of each player is now the total transmit power over the $N$ subchannels and the $M$ time slots

$$u_q(p_q) = \sum_{k=1}^{N} \sum_{m=1}^{M} p_{k,m}^q.$$  (30)

The constraint now is that the expected rate, conditioned to the observation on the first time slot, has to be no smaller than a given value. The feasible set is now

$$\tilde{F}_q(p_{-q}) = \{p_q \in \mathbb{R}^{NM \times 1} : \tilde{R}_q(p_q, p_{-q}) \geq \bar{R}_q^0, 0 \leq p_{k,m}^q \leq p_{q}^{\max}(k), \forall k = 1, \ldots, N, m = 1, \ldots, M\}$$  (31)

and the game is then

$$\tilde{G}_2 = \{\Omega, \{\tilde{F}_q(p_{-q})\}_{q \in \Omega}, \{u_q(p_q)\}_{q \in \Omega}\}.$$  (32)

The optimal strategy for each player amounts to solving the following optimization problem

$$\hat{P}_2 \quad \min_{p_q} u_q(p_q) \quad \forall q \in \Omega$$

subject to $p_q \in \tilde{F}_q(p_{-q})$
where the set \( \tilde{F}_q(p_{-q}) \), given the power vector \( p_{-q} \) of the other players, is a convex set. The minimization problem in (33) for each player \( q \), given the strategies of the others, is then a convex optimization problem, since the objective function is a linear (then convex) function of \( p_q \). The solution can be written in closed form \( p_q^* = g(p_{-q}) \), where (see Appendix C)

\[
[g(p_{-q})]_{k,m} = \left[ \frac{b^q(k,m) + \sqrt{b^q(k,m)^2 - 4a^q(k,m)c^q(k,m)}}{2a^q(k,m)} \right] p_{q_{max}}^{max}(k)_{0} \forall k = 1, \ldots, N, m = 1, \ldots, M
\]

with

\[
a^q(k,m) = a^q_n(k,m) a^q_I(k,m)
\]

\[
b^q(k,m) = a^q_n(k,m) + a^q_I(k,m) - \lambda_q a^q_n(k,m) a^q_I(k,m)
\]

\[
c^q(k,m) = 1 - \lambda_q [a^q_n(k,m) \beta_{k,m} + a^q_I(k,m) \gamma_{k,m}]
\]

where the Lagrange multiplier \( \lambda_q \) must satisfy the rate constraint \( \bar{R}_q(p_q^*, p_{-q}) = R_{q_q}^0 \). However, each user’s feasible set is not jointly convex with respect to the power vectors of all the users, i.e. it is not convex in \( p = (p_q)_{q=1}^Q \). This makes its study much harder than the standard Nash Equilibrium Problem (NEP). Nevertheless, as with the min power game, game \( \tilde{G}_2 \) is a Generalized Potential Game. In particular, the existence of a NE of the potential game can be proved directly by the existence of a maximum of the potential function \( \Phi \) on the set \( \tilde{X} \) of the game. To exploit the theory of GPG, we must prove that game \( \tilde{G}_2 \) admits a non-empty feasible set. The proof of this result is in Appendix D, containing the sufficient conditions under which the feasible set of the game \( \tilde{G}_2 \), i.e.

\[
\tilde{X} = \{ p \in \mathbb{R}^{NMQ} : \tilde{R}_q(p) \geq R_{q_q}^0, 0 \leq p_{k,m}^q \leq p_{q_{max}}^q(k), \forall k, m, \forall q \in \Omega \}
\]

is compact and non-empty. Clearly, as with game \( G_2 \), game \( \tilde{G}_2 = \{ \Omega, \{ \tilde{F}_q(p_{-q}) \}_{q \in \Omega}, \{ u_q(p_q) \}_{q \in \Omega} \} \) is a Generalized Potential Game (GPG) with potential function \( \Phi(p) \) the sum of the objective functions of all players, i.e. \( \Phi(p) = \sum_{q=1}^Q u_q(p_q) \). Since game \( \tilde{G}_2 \) is a GPG, a NE could be computed by solving in a centralized way the following optimization problem

\[
\min \limits_{p} \Phi(p) \quad \text{subject to } p \in \tilde{X}
\]
The existence of a GNE for the game $\tilde{\mathcal{G}}_2$ can then be deduced from the existence of a minimum point for the function $\Phi(p)$ over the set $\tilde{\mathcal{X}}$, since the sufficient conditions [19, 20] for the existence of a NE are: i) the compactness of the set $\tilde{\mathcal{X}}$ (as reported in Appendix D); and ii) the upper semicontinuity of $\Phi$ on $\tilde{\mathcal{X}}$ (it follows by the continuity of the potential function $\Phi(p)$).

After having proved that game $\tilde{\mathcal{G}}_2$ falls in the class of potential games, it remains open the problem of choosing a convergent algorithm which reaches the Nash equilibria of the game. The direct solution of problem (37) could require, in general, a centralized approach. Nevertheless, decomposition methods can be used for the derivation of the Nash equilibrium of a GPG [15].

Since every player is trying to minimize its own objective function, it is natural to use an iterative algorithm, such as a Gauss-Seidel best-response algorithm, where at each step, every player finds its optimal strategy, given the strategies of the others. One of the most critical aspects in the use of these iterative methods is the proof of convergence. To overcome this difficulty, a Regularized Gauss-Seidel Algorithm (RGSA) for GPGs has been proposed in [15] which essentially differs from the standard Gauss-Seidel for a regularization term which is added to the objective function in order to ensure and prove the convergence of the algorithm. The RGSA applied to game $\tilde{\mathcal{G}}_2$ leads to the algorithm outlined in Table 3. We should note that the RGSA is well defined since all subproblems in each step S.2 in Table 3 always admit a unique solution. As far as the convergence of Algorithm 3 is concerned, in Appendix E we report the sufficient conditions under which the regularized Gauss-Seidel Algorithm applied to the game $\tilde{\mathcal{G}}_2$ in (32) converges to a NE. Finally, in Figs. 7 and 8 we report the simulation results corresponding to our proposed minimum power games, under constraint on the expected rate: Fig. 7 refers to the min power game, with no pricing, while Fig. 8 refers to the game including pricing. The simulations in Fig. 7 and 8 are computed over 100 independent Markov chains with idle-to-idle transition probability $\omega_k = \omega = 0.1$ and busy-to-busy rate $\mu_k = \mu = 0.2$. The number of subcarriers is set to 24 while the expected target rate in both cases is set to 3 bits per symbol. From both Figs. 7 and 8 we can verify that the simple statistical knowledge of the
Algorithm 3: Regularized Gauss-Seidel algorithm

S.0: Choose any feasible power allocation \( p^0 = (p^0_q, \ldots, p^0_Q) \),

a positive regularization parameter \( \tau > 0 \) and set iteration index \( n = 0 \)

S.1: If \( p_q(n) \) satisfies a suitable termination then STOP, otherwise

S.2: For \( q = 1, \ldots, Q \) compute a solution \( p_q(n+1) \) of

\[
\begin{align*}
\min_{p_q} & \quad u_q(p_q) + \tau \| p_q - p_q(n) \|^2 \\
\text{s.t.} & \quad p_q \in \mathcal{P}_q(p_1(n+1), \ldots, p_{q-1}(n+1), p_{q+1}(n), \ldots, p_Q(n))
\end{align*}
\]

S.3 Set \( p(n+1) = (p_1(n+1), \ldots, p_Q(n+1)) \), \( n = n + 1 \) and go to S.1.

Table 3: Regularized Gauss-Seidel Algorithm

transition rates yields performance close to the idealistic case where the interference activity is non-causally known. In both cases, the advantage of the statistical approach with respect to the case where there is no knowledge is considerable. Finally, we can notice, by comparing Figs. 7 and 8 the advantage resulting from the introduction of pricing (around 3 dB).

4 Conclusion

In this paper we have shown alternative game-theoretic algorithms suitable for femtocell networks because of their fast convergence, assuming either pure competition among FAPs or local coordination through the exchange of pricing parameters through the backhaul. We have devoted special attention to modeling the interference activity as a Markov model. It is worth investigating the generalization of this work to cases where the interference activity is modeled with more complicated statistical laws, taking into account the geographic distribution of FAPs.

Appendix A

Let us consider the feasible set of \( (P_C) \), i.e.

\[
\mathcal{X} = \{ p \in \mathbb{R}^{NQ \times 1} : R_q(p) \geq R_q^0, 0 \leq p^q_k \leq p^{\max}_q(k), \forall q \in \Omega, \forall k = 1, \ldots, N \}.
\] (38)
Let us prove that the feasible set $X$ of $(P_C)$ is a nonempty, closed and bounded (then compact) subset of $\mathbb{R}^{NQ \times 1}$ if the matrices $A_k$ defined in (43) are P-matrices, for all $k = 1, \ldots, N$.

Sufficient conditions for which this happens are

$$\sum_{r \in N_q} \frac{|H_{k}^{rq}|^2}{|H_{k}^{qq}|^2} < \frac{1}{e^{R^0_q} - 1} \quad \forall q \in \Omega, \; \forall k = 1, \ldots, N. \quad (39)$$

Let us start by considering the constraints $R_q(p) \geq R^0_q$ or

$$\sum_{k=1}^{V_N^q} \log \left(1 + p_k^q a_n^q(k)\right) \geq R^0_q, \quad \forall q \in \Omega$$

where the coefficient $a_n^q(k) = \frac{|H_{k}^{qq}|^2}{\sigma_q^2 + \sum_{r \in N_q} p_r^k |H_{k}^{rq}|^2}$, while $V_N^q \subseteq \{1, \ldots, N\}$, is the subset of subcarriers that the player $q$ is using during the game. Observe that (40) is surely valid if we prove $\forall k \in V_N^q$ that

$$\log \left(1 + \frac{p_k^q |H_{k}^{qq}|^2}{\sigma_q^2 + \sum_{r \in N_q} p_r^k |H_{k}^{rq}|^2}\right) \geq R^0_q$$

so that we have to verify $\forall k \in V_N^q$ the following set of inequalities

$$p_k^q |H_{k}^{qq}|^2 - (e^{R^0_q} - 1) \sum_{r \in N_q} p_r^k |H_{k}^{rq}|^2 \geq (e^{R^0_q} - 1)\sigma_q^2 \quad \forall q \in \Omega. \quad (42)$$

Defining the vector $p_k = (p_k^q)_{q=1}^Q$ and the $Q \times Q$-dimensional matrices

$$A_k = \begin{bmatrix}
|H_{k}^{11}|^2 & -(e^{R^0_q} - 1)|H_{k}^{12}|^2 & \ldots & -(e^{R^0_q} - 1)|H_{k}^{1Q}|^2 \\
-(e^{R^0_q} - 1)|H_{k}^{21}|^2 & |H_{k}^{22}|^2 & \ldots & -(e^{R^0_q} - 1)|H_{k}^{2Q}|^2 \\
\vdots & \vdots & \ddots & \vdots \\
-(e^{R^0_q} - 1)|H_{k}^{Q1}|^2 & -(e^{R^0_q} - 1)|H_{k}^{Q2}|^2 & \ldots & |H_{k}^{QQ}|^2
\end{bmatrix}^T \quad (43)$$

we can express the system of inequalities in (42) as

$$A_k p_k \geq v_k \quad \forall k \in V_N^q, \quad (44)$$

where the positive entries of the vector $v_k = (v_k^q)_{q=1}^Q$ are given by $v_k^q = (e^{R^0_q} - 1)\sigma_q^2$. It can be observed that each matrix $A_k$ is a Z-matrix, i.e. a matrix with all off-diagonal elements
nonpositive. Furthermore, if we ensure that each $A_k$ is also a P-matrix, i.e. a matrix whose determinant and all principal minors are positive [21], we can deduce that its inverse is well defined. By imposing diagonally dominance on the elements of the matrices $A_k, \forall k = 1, \ldots, N$, we can find the following sufficient conditions for them to be P-matrices, i.e.

$$
\sum_{r \in \mathcal{N}_q} \frac{|H_{kq}^r|^2}{|H_{kk}^q|^2} < \frac{1}{e^{R_q^0} - 1} \quad \forall q \in \Omega, \forall k = 1, \ldots, N.
$$

(45)

Hence, by considering the general case $\mathcal{V}_N^q = \{1, \ldots, N\}$, we can deduce from (44) that there exist positive vectors $p_k, p_{\max}^k(k) \triangleq (p_{q}^{\max}(k))_{q=1}^Q \in \mathbb{R}_+^Q$ such that

$$
p_{\max}^k(k) > p_k \geq A_k^{-1}v_k \quad \forall k = 1, \ldots, N,
$$

(46)
or the sets $\mathcal{D}_k = \{p_k \in \mathbb{R}_+^Q : \log(1+p_k^q a_n^q(k)) \geq R_q^0, 0 \leq p_k^q \leq p_{q}^{\max}(k), \forall q \in \Omega\}$ are non empty. Of course also the product set $\mathcal{D} = \{\prod_{k=1}^N \mathcal{D}_k\} \subseteq \mathcal{X}$ is non empty, so that the nonemptiness of $\mathcal{X}$ is implied. Furthermore, since $\forall q$ the set $\{p \in \mathbb{R}_+^{NQ \times 1} : R_q(p) \geq R_q^0\}$ is the upper level set of the continuous function $R_q(p)$, then it is closed for all scalar $R_q^0$ [22]. It follows that the set $\mathcal{X} = \{p \in \mathbb{R}_+^{NQ \times 1} : R_q(p) \geq R_q^0, 0 \leq p_k^q \leq p_{q}^{\max}(k), \forall q \in \Omega, \forall k = 1, \ldots, N\}$, as non empty intersection of closed sets is closed [22] and, since it is also bounded, its compactness has been proved.

**Appendix B**

*Convergence Analysis of MADP Algorithm.*

Proceeding as in [12], in order to prove the convergence of the algorithm it is sufficient to show that

a) With a proper choice of the step $\alpha_q(n)$, MADP converges to a fixed point;

b) This point is a solution of the KKT conditions of the modified game in (22) and then it
is also a solution point of the optimization problem

$$\max_{\boldsymbol{p}} \bar{\bar{R}}(\boldsymbol{p})$$

s.t. \( \boldsymbol{p} \in \tilde{\mathcal{P}} \) \hspace{1cm} (47)

where the FAPs’ sum rate is \( \bar{\bar{R}}(\boldsymbol{p}) = \sum_{q=1}^{Q} \bar{R}_q(\boldsymbol{p}) \) and \( \tilde{\mathcal{P}} \) is the cartesian product of the sets \( \tilde{\mathcal{P}}_q \).

Let us denote with \( U_1(n) = \bar{\bar{R}}(\boldsymbol{p}(n)) \) the sum utility reached at the step \( n \) of the MADP algorithm. Then for each user \( q \) we must prove that there exists a sequence \( \alpha_q(n) > 0 \) so that \( U_1(n) \) is monotonically increasing and convergent, i.e., \( U_1(n+1) \geq U_1(n) \forall n \) and \( U_1(n) \to U_1^* \) as \( n \to \infty \). As discussed in [12], we only need to show that \( U_1(n) \) is monotonically increasing, i.e. it suffices to consider a given iteration \( n \) in which user \( q \) is selected to update its power profile, and show that \( U_1(\boldsymbol{p}_q(n+1)) \geq U_1(\boldsymbol{p}_q(n)) \), where the total utility \( U_1 \) is now regarded as a function of \( \boldsymbol{p}_q \) because only the power profile of user \( q \) is updated. Hence our goal is prove that \( U_1(\boldsymbol{p}_q(n+1)) \geq U_1(\boldsymbol{p}_q(n)) \). To do this we will use the descent lemma to bound \( U_1(\boldsymbol{p}_q(n+1)) \).

Descent lemma [12] says that if a function \( F : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable and its gradient is Lipschitz continuous with Lipschitz constant equal to \( K \) then, \( \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \)

$$F(\boldsymbol{x} + \boldsymbol{y}) \leq F(\boldsymbol{x}) + \boldsymbol{y}^T \nabla F(\boldsymbol{x}) + \frac{K}{2} \| \boldsymbol{y} \|_2^2 . \hspace{1cm} (48)$$

One sufficient condition for Lipschitz continuity is that the \( l_2 \)-norm of the Hessian matrix of \( F(\boldsymbol{x}) \) is bounded, in which case this bound can be used for the Lipschitz constant. It can be easily shown that it is true for \( U_1(\boldsymbol{p}_q) \). Specifically, there exists a constant \( B_{U_1}^q \) which upper bounds the \( l_2 \)-norm of the Hessian matrix of \( U_1(\boldsymbol{p}_q) \) independent of others’ power profiles.

Applying the Descent Lemma to \(-U_1(\boldsymbol{p}_q)\), we get

$$U_1(\boldsymbol{p}_q(n+1)) \geq U_1(\boldsymbol{p}_q(n)) + [\boldsymbol{p}_q(n+1) - \boldsymbol{p}_q(n)]^T \nabla \boldsymbol{p}_q U_1(\boldsymbol{p}_q(n)) - \frac{B_{U_1}^q}{2} \| \boldsymbol{p}_q(n+1) - \boldsymbol{p}_q(n) \|_2^2 . \hspace{1cm} (49)$$

Hence to prove that \( U_1(\boldsymbol{p}_q(n+1)) \geq U_1(\boldsymbol{p}_q(n)) \), it suffices to show that

$$[\boldsymbol{p}_q(n+1) - \boldsymbol{p}_q(n)]^T \nabla \boldsymbol{p}_q U_1(\boldsymbol{p}_q)(n) \geq \frac{B_{U_1}^q}{2} \| \boldsymbol{p}_q(n+1) - \boldsymbol{p}_q(n) \|_2^2 . \hspace{1cm} (50)$$
Using the power updating rule

\[ p_h^q (n + 1) = p_h^q (n) + \alpha_q (n) (p_h^q - p_h^q (n)) \quad (51) \]

with the best response of user q defined in (29), the inequality in (50) can be written as

\[
\left[ p_q^* - p_q (n) \right]^T \nabla_p U_1 (p_q (n)) \geq \alpha_q (n) \frac{B_{U_1}}{2} \| p_q^* - p_q (n) \|_2^2 .
\] (52)

Observe that

\[
\frac{\partial U_1 (p_q)}{\partial p_h^q} \bigg|_{p_q = p_q (n)} = \frac{\beta_h^q SNIR_h^q}{1 + SNIR_h^q} \cdot \frac{1}{p_h^q (n)} + \frac{\gamma_h^q SNIR_h^q}{1 + SNIR_h^q} \cdot \frac{1}{p_h^q (n)} - \sum_{r \in \mathcal{N}_q} \pi_h^r (n) |H_h^{qr}|^2 ,
\] (53)

then exploiting the result in (29), we can write the left hand side (LHS) of (52) as

\[
\text{LHS} = \sum_{q=1}^{Q} \frac{\sum_{r \in \mathcal{N}_q} \pi_h^r (n) |H_h^{qr}|^2}{2p_h^q (n)} (p_h^q - p_h^q (n))^2 + \sum_{q=1}^{Q} \frac{c_h^q (\eta_h^q - \nu_h^q) (p_h^q - p_h^q (n))}{p_h^q (n)} \left( \sum_{r \in \mathcal{N}_q} \pi_h^r (n) |H_h^{qr}|^2 + \nu_h^q - \eta_h^q \right) .
\] (54)

Now from (54), with the same steps as in [12], to ensure that

\[
\text{LHS} \geq \alpha_q (n) \frac{B_{U_1}}{2} \| p_q^* - p_q (n) \|_2^2 ,
\] (55)

we can choose the step \( \alpha_q (n) \) as

\[
\alpha_q (n) \leq \min \left\{ \frac{2A^n_q}{B_{U_1}}, 1 \right\}
\] (56)

where the coefficient \( A^n_q \) is defined as

\[
A^n_q = \min_h \left\{ \frac{\sum_{r \in \mathcal{N}_q} \pi_h^r (n) |H_h^{qr}|^2}{p_h^q (n) c_h^q} \right\} .
\] (57)

Now in order to prove the point b), let \( U_1^* \) a fixed point of the algorithm such that \( U_1 (n) = U_1^* \) for some index n. Then since this is a fixed point, it follows that \( p_h^q (n) = p_h^{qr}, \forall h, q \). It can then be seen that for all q, \( p_h^q \) must be an optimal solution to the problem (22), given the other users current power profiles and interference price vectors. Hence, \( p(n) \) will satisfy also the KKT conditions of the problem (47).
Appendix C

In order to find the optimal solutions of the convex problem \( \tilde{P}_2 \) by studying the Lagrange dual problem, some additional constraint qualification conditions must hold, beyond convexity, to ensure strong duality [23]. One simple constraint qualification is Slater’s condition, i.e. we must verify that some strictly feasible point exists. We can prove that the set \( \tilde{F}_q(p_q) \) for each user \( q \) fixed the strategies of the others is nonempty. More specifically, the constraint \( R_q(p) > R_q^0 \) can be written as

\[
\sum_{k=1}^{\mathcal{V}_N} \sum_{m=1}^{\mathcal{V}_M} [\beta_{k,m} \log (1 + p_{k,m}^q a_n^q(k,m)) + \gamma_{k,m} \log (1 + p_{k,m}^q a_l^q(k,m))] > R_q^0
\]

where we have denoted with \( \mathcal{V}_N^q \subseteq \{1, \ldots, N\} \) and \( \mathcal{V}_M^q \subseteq \{1, \ldots, M\} \) the subsets, respectively, of subcarriers and time slots that the player \( q \) is using during the game. Since \( a_n^q(k,m) > a_l^q(k,m) \), to verify (58), it is sufficient to prove that

\[
\log (1 + p_{k,m}^q a_l^q(k,m)) > R_q^0, \quad \forall \ k \in \mathcal{V}_N^q, \ m \in \mathcal{V}_M^q, \ q \in \Omega
\]

and clearly it exists always a set of positive values \( p_{k,m}^q, p_{k,m}^{\text{max}}(k) \in \mathbb{R}_+ \) such that

\[
p_{k,m}^{\text{max}}(k) > p_{k,m}^q \geq (e^{R_q^0} - 1) \frac{1}{a_l^q(k,m)} \quad \forall \ k \in \mathcal{V}_N^q, \ m \in \mathcal{V}_M^q, \ q \in \Omega.
\]

Let us consider, for \( k = 1, \ldots, N, \ m = 1, \ldots, M \), the KKT conditions of the optimization problem \( \tilde{P}_2 \):

\[
1 - \lambda_q \left[ \frac{\beta_{k,m} a_n^q(k,m)}{1 + p_{k,m}^q a_n^q(k,m)} + \frac{\gamma_{k,m} a_l^q(k,m)}{1 + p_{k,m}^q a_l^q(k,m)} \right] - \mu_{k,m}^q + \alpha_{k,m}^q = 0
\]

\[
0 \leq \lambda_q \perp \tilde{R}_q(p_q, p_{-q}) - R_q^0 \geq 0
\]

\[
0 \leq p_{k,m}^q \perp \mu_{k,m}^q \geq 0
\]

\[
0 \leq \alpha_{k,m}^q \perp p_{k,m}^{\text{max}}(k) - p_{k,m}^q \geq 0
\]

Observe that if \( p_{k,m}^{\text{max}}(k) - p_{k,m}^q > 0 \), then \( \alpha_{k,m}^q = 0 \) so that, by eliminating in (61) the multiplier \( \mu_{k,m}^q \), we obtain

\[
0 \leq \left[ 1 - \lambda_q \left( \frac{\beta_{k,m} a_n^q(k,m)}{1 + p_{k,m}^q a_n^q(k,m)} + \frac{\gamma_{k,m} a_l^q(k,m)}{1 + p_{k,m}^q a_l^q(k,m)} \right) \right] \perp p_{k,m}^q \geq 0
\]

\[
0 \leq \lambda_q \perp \tilde{R}_q(p_q, p_{-q}) - R_q^0 \geq 0
\]
where $\lambda_q > 0$ otherwise complementarity yields $p_{k,m}^q = 0$, $\forall k = 1, \ldots, N, m = 1, \ldots, M$, and the rate constraint is contradicted. Then the optimum power vector must satisfy the following equation

$$a^q(k, m)(p_{k,m}^q)^2 + b^q(k, m)p_{k,m}^q + c^q(k, m) = 0 \quad (63)$$

having set

$$a^q(k, m) = a_n^q(k, m) a_q^q(k, m)$$
$$b^q(k, m) = a_n^q(k, m) + a_q^q(k, m) - \lambda_q a_n^q(k, m)a_q^q(k, m) \quad (64)$$
$$c^q(k, m) = 1 - \lambda_q [a_n^q(k, m)\beta_{k,m} + a_q^q(k, m)\gamma_{k,m}]$$

The solutions of (63) are

$$p_{k,m}^q = \begin{cases} 
-\frac{b^q(k, m) - \sqrt{b^q(k, m)^2 - 4a^q(k, m)c^q(k, m)}}{2a^q(k, m)} & \text{for } \lambda_q > 1 \\
-\frac{b^q(k, m) + \sqrt{b^q(k, m)^2 - 4a^q(k, m)c^q(k, m)}}{2a^q(k, m)} & \text{for } \lambda_q \leq 1
\end{cases} \quad (65)$$

It can be proved that $\forall \lambda_q > 0$, it results $p_{k,m}^q \leq 0$, $b^q(k, m)^2 - 4a^q(k, m)c^q(k, m) \geq 0$, and

$$p_{k,m}^b = \begin{cases} 
> 0 & \text{for } \lambda_q > \frac{1}{a_n^q(k, m)\beta_{k,m} + a_q^q(k, m)\gamma_{k,m}} \\
\leq 0 & \text{for } \lambda_q \leq \frac{1}{a_n^q(k, m)\beta_{k,m} + a_q^q(k, m)\gamma_{k,m}}
\end{cases} \quad (66)$$

According to the above considerations the solution is $p_{k,m}^q = p_{k,m}^b$ for $0 < p_{k,m}^q < p_{q}^{\max}(k)$ and $\lambda_q > \frac{1}{a_n^q(k, m)\beta_{k,m} + a_q^q(k, m)\gamma_{k,m}}$ so that we can write the optimal power allocation vector as

$$p_{k,m}^{q^*} = \lfloor p_{k,m}^b, p_{q}^{\max}(k) \rfloor \quad (67)$$

where the multiplier $\lambda_q$ is chosen in order to satisfy the constraint $\tilde{R}_q(p^*, p_{-q}) = R_q^0$.

**Appendix D**

*Proof that the feasible set of the game $\tilde{G}_2$ is compact and non-empty so that it can be cast as a GPG.*

Let us start by the following definition of Generalized Potential Game given in [15]:
Definition 1 A GNEP is a Generalized Potential Game if

a) There exists a nonempty, closed set $\tilde{X} \subseteq \mathbb{R}^n$ such that

$$X_q(x_{-q}) = \{x_q \in D_q : (x_q, x_{-q}) \in \tilde{X}\} \quad \forall q = 1, \ldots, Q$$

(68)

where $D_q \subseteq \mathbb{R}^{n_q}$ are nonempty, closed sets such that $\prod_{q=1}^{Q} D_q \cap \tilde{X} \neq \emptyset$;

b) There exists a continuous function, potential function, $\Phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, named potential function, such that $\forall q \in \Omega$, $\forall x_{-q}$ and for all $y_q, z_q \in X_q(x_{-q})$

$$u_q(y_q, x_{-q}) - u_q(z_q, x_{-q}) > 0$$

(69)

implies

$$\Phi(y_q, x_{-q}) - \Phi(z_q, x_{-q}) \geq u_q(y_q, x_{-q}) - u_q(z_q, x_{-q})$$

(70)

where $u_q$ is the $q$-th player payoff function.

According to the Definition 1, we have to check the effectiveness of the conditions a) and b) for the game $\tilde{G}_2$. As regard the condition a), let us consider the feasible set of the game $\tilde{G}_2$, i.e.

$$\tilde{X} = \{ p \in \mathbb{R}^{NMQ} : R_q(p) \geq R^0_q, 0 \leq p^q_{k,m} \leq p_{max}^q(k), \forall k \in \{1, \ldots, N\}, \forall m \in \{1, \ldots, M\}, \forall q \in \Omega \}. $$

(71)

Then we have to prove the following Lemma.

Lemma 1 The feasible set $\tilde{X}$ of the game $\tilde{G}_2$ is a nonempty, closed and bounded (then compact) subset of $\mathbb{R}^{NMQ}$ if the matrices $A_k$ defined in (76) are P-matrices, for all $k = 1, \ldots, N$.

Sufficient conditions for which this happens are

$$\sum_{r \in N_q} \frac{|H^r_q|^2}{|H^p_q|^2} < \frac{1}{e^{R^0_q} - 1} \quad \forall q \in \Omega, \quad \forall k = 1, \ldots, N .$$

(72)

---

5We assume that $\prod_{q=1}^{Q} \mathbb{R}^{n_q} = \mathbb{R}^n$. 

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Proof. Let us start by considering the constraints \( \tilde{R}_q(p) \geq R^0_q \) that, by considering only the subcarriers and the time slots that are effectively occupied, can be written as

\[
\sum_{k=1}^{V^q_N} \sum_{m=1}^{V^q_M} \left[ \beta_{k,m} \log \left( 1 + p_{k,m}^q a_n^q(k,m) \right) + \gamma_{k,m} \log \left( 1 + p_{k,m}^q a_j^q(k,m) \right) \right] \geq R^0_q, \quad \forall q \in \Omega \tag{73}
\]

where \( V^q_N \subseteq \{1, \ldots, N\} \), \( V^q_M \subseteq \{1, \ldots, M\} \) are the subsets, respectively, of subcarriers and time slots, that the player \( q \) is using during the game. We can note that \( a_n^q(k,m) > a_j^q(k,m) \) then (73) is surely valid if we prove that

\[
\log \left( 1 + \frac{p_{k,m}^q |H_k^{qq}|^2}{\sigma^2_{n,q}(k) + \sum_{r \in N_q} p_{k,m}^r |H_k^{rq}|^2 + \sigma^2_{j,r}(k,m)} \right) \geq R^0_q \tag{74}
\]

so that we have to verify \( \forall k \in V^q_N, \ m \in V^q_M \) the following set of inequalities

\[
p_{k,m}^q |H_k^{qq}|^2 - (e^{R^0_q} - 1) \sum_{r \in N_q} p_{k,m}^r |H_k^{rq}|^2 \geq (e^{R^0_q} - 1)(\sigma^2_{n,q}(k) + \sigma^2_{j,r}(k,m)) \quad \forall q \in \Omega. \tag{75}
\]

Defining the vector \( p_{k,m} = (p_{k,m}^q)_{q=1}^Q \) and the matrices

\[
A_k = \begin{bmatrix} |H_k^{11}|^2 & -(e^{R^0_q} - 1)|H_k^{12}|^2 & \ldots & -(e^{R^0_q} - 1)|H_k^{1Q}|^2 \\ -(e^{R^0_q} - 1)|H_k^{21}|^2 & |H_k^{22}|^2 & \ldots & -(e^{R^0_q} - 1)|H_k^{2Q}|^2 \\ \vdots & \vdots & \ddots & \vdots \\ -(e^{R^0_q} - 1)|H_k^{Q1}|^2 & -(e^{R^0_q} - 1)|H_k^{Q2}|^2 & \ldots & |H_k^{QQ}|^2 \end{bmatrix}^T \tag{76}
\]

we can express the system of inequalities in (75) as

\[
A_k p_{k,m} \geq v_{k,m} \quad \forall k \in V^q_N, \ m \in V^q_M \tag{77}
\]

where the positive entries of the vector \( v_{k,m} = (v_{k,m}^q)_{q=1}^Q \) are given by \( v_{k,m}^q = (e^{R^0_q} - 1)(\sigma^2_{n,q}(k) + \sigma^2_{j,r}(k,m)) \). Proceeding as in Appendix A, it can be observed that each matrix \( A_k \) is a Z-matrix and if we ensure that each \( A_k \) is also a P-matrix [21], we can deduce that its inverse is well defined. By imposing diagonally dominance on the elements of the matrices \( A_k, \forall k = 1, \ldots, N \), we can find the following sufficient conditions for them to be P-matrices, i.e.

\[
\sum_{r \in N_q} \frac{|H_k^{rq}|^2}{|H_k^{qq}|^2} < \frac{1}{e^{R^0_q} - 1} \quad \forall q \in \Omega, \forall k = 1, \ldots, N. \tag{78}
\]
Hence, by considering the general case \( \mathcal{V}_N = \{1, \ldots, N\} \) and \( \mathcal{V}_M = \{1, \ldots, M\} \), we can deduce from (77) that there exist positive vectors \( \mathbf{p}_{k,m}, \mathbf{p}^{\text{max}}(k) \triangleq (p^{\text{max}}_q(k))_{q=1}^Q \in \mathbb{R}^Q \) such that

\[
\mathbf{p}^{\text{max}}(k) > \mathbf{p}_{k,m} \geq A^{-1}v_{k,m} \quad \forall \ k = 1, \ldots, N, \ m = 1, \ldots, M
\]

(79)
or the sets \( \mathcal{F}_{k,m} = \{ \mathbf{p}_{k,m} \in \mathbb{R}^Q : \log(1 + p^{\theta}_q(k,m)q^q(k,m)) \geq R^0_q, \ 0 \leq p^{\theta}_q(k,m) \leq p^{\text{max}}_q(k), \ \forall q\} \) are non-empty. Of course also the product set \( \mathcal{F} = \{ \prod_{k=1}^N \prod_{m=1}^M \mathcal{F}_{k,m} \} \subseteq \bar{\mathcal{X}} \) is non-empty, so that the nonemptiness of \( \bar{\mathcal{X}} \) is implied.

Furthermore, \( \forall q \in \Omega \) the set \( \{ \mathbf{p} \in \mathbb{R}^{MNQ} : \bar{R}_q(\mathbf{p}) \geq R^0_q \} \), is the upper level set of the continuous function \( \bar{R}_q(\mathbf{p}) \), then it is closed for all scalar \( R^0_q \) [22]. Hence the set \( \bar{\mathcal{X}} = \{ \mathbf{p} \in \mathbb{R}^{MNQ} : \bar{R}_q(\mathbf{p}) \geq R^0_q, \ 0 \leq p^{\theta}_q(k,m) \leq p^{\text{max}}_q(k), \ \forall k \in \{1, \ldots, N\}, \ \forall m \in \{1, \ldots, M\}, \ \forall q \in \Omega \} \), as non empty intersection of closed sets, is closed [22] and, since it is also bounded, its compactness is proved.

Verification of condition b) is rather straightforward. In our case the objective functions do not depend on the other players variables, i.e. \( u_q(\mathbf{p}_q, \mathbf{p}_{-q}) = u_q(\mathbf{p}_q) \) so that the interaction of the players takes places only at the level of feasible sets. In this case, it is immediate to see that condition b) is satisfied with the potential function \( \Phi(\mathbf{p}) \) simply given by the sum of the objective functions of all players, i.e.

\[
\Phi(\mathbf{p}) = \sum_{q=1}^Q u_q(\mathbf{p}_q)
\]

(80)
and this concludes the proof.

\[\square\]

**Appendix E**

*Convergence of the RGSA.*

Let us prove under which conditions the regularized Gauss-Seidel Algorithm applied to the game \( \tilde{\mathcal{G}}_2 \) converges to a NE. For this purpose we can exploit the following Theorem stated in [15] for generalized potential games.

**Theorem 1** Assume that for each player \( q \)
a) The objective function $u_q(p_q)$ is continuous on $\tilde{X}$;

b) The feasible sets $\bar{F}_q(\cdot)$ are inner semicontinuous relative to $\text{dom}(\bar{F}_q(\cdot))$ \(^6\).

c) The objective function $u_q(\cdot, p_q)$ and the feasible sets $\bar{F}_q(p_q)$ are convex.

Let $\{p^k\}$ be the sequence generated by Algorithm 3 and let $p^*$ be a cluster point of this sequence. Then $p^*$ is a Nash equilibrium of game $\tilde{G}_2$.

Then for the game $\tilde{G}_2$ the following theorem can be proved.

**Theorem 2** The regularized Gauss-Seidel Algorithm applied to the game $\tilde{G}_2$ converges to a NE if the sufficient conditions in (72) hold.

**Proof.** We have proved in Appendix D that under the sufficient conditions in (72) the game $\tilde{G}_2$ is a GPG. Then for the convergence of the RGSA to a NE we have just to verify the assumptions of Theorem 1. Observe that conditions a) and c) are verified since $u_q(p_q) = \sum_{k=1}^{N} \sum_{m=1}^{M} p_{k,m}^q$ is continuous on $\tilde{G}_2$, and the feasible sets $\bar{F}_q(p_q)$, fixed $p_{-q}$, are convex $\forall q$. As regard the condition b), the inner semicontinuity requirement says that if $p^*$ belongs to $\tilde{X}$ and we take any sequence $\{p^k_{-q} \subset \text{dom}(\bar{F}_q)\}$ such that $\{p^k_{-q}\} \to p^*_{-q}$, for every $q$, then we may find points $p^k_q \in \bar{F}_q(p^k_{-q})$ such that $\{p^k_q\} \to p^*_q$. It can be noted that the set $\tilde{X}$ is bounded and generated by continuous functions then the inner semicontinuity of each player set $\bar{F}_q(p_{-q})$ can be easily deduced.

Finally, we must prove the existence of a cluster point, i.e. of a point $p^*$ of $\{p^k\}$ for which it exists a subsequence $\{p^{k_i}\} \subset \{p^k\}$ such that $\{p^{k_i}\} \to p^*$. Observing that the set $\tilde{X}$ is bounded, we can deduce that a cluster point exists, since, from the Bolzano-Weirestrass theorem [24], every bounded sequence in $\mathbb{R}^{MNQ \times 1}$ has a cluster point.

**References**


\[^6\]The $\text{dom}(\bar{F}_q(\cdot))$ is the set of points $p_{-q}$ for which $\bar{F}_q(p_{-q})$ is non empty.


Figure 1: Femtocell network scenario.

Figure 2: Percentage number of conflicts vs. number of subcarriers for different number of FAPs.
Figure 3: Sum of the radiated power vs. iteration index for the min-power game with and without pricing.

Figure 4: Sum rate vs. iteration index for the min-power game with and without pricing.
Figure 5: Sum rate of two FAPs vs. number of time slots for the maximum expected rate game without pricing.

Figure 6: Sum rate of two FAPs vs. number of time slots for the maximum expected rate game with pricing.
Figure 7: Sum power of two FAPs vs. number of time slots for the min-power game with Markovian interference (no pricing).

Figure 8: Sum power of two FAPs vs. number of time slots for the min-power game with Markovian interference and with pricing.