Kalman-like Filtering and Smoothing for Reciprocal Sequences

E. Baccarelli, R. Cusati, G. Di Blasio
INFOCOM Dpt., University of Rome 'La Sapienza', Via Eudossiana 18, 00184 Roma, Italy

Abstract - The MMSE filtering problem of reciprocal Gaussian sequences in additive white Gaussian noise is solved in a recursive and causal form. The solution, based on the innovations method, is expressed in terms of a set of recursive equations formally similar to those of the well-known Kalman filter; it gives as by-product the solution of the MMSE smoothing problem (fixed-point, fixed-interval, fixed-lag). The performance of the proposed estimators is given by recursive expressions.

I. MODELS

Gauss-Markov random processes are based on a causal recursive generation mechanism where each data value depends on the previously generated samples. In some practical situations, however, the causality assumption no longer true and a better modeling is often offered by the reciprocal processes [1,2].

In this work the general problem of the recursive least squares estimation (filtering and smoothing) of a multivariate discrete-index reciprocal Gaussian process (RGP) in additive white Gaussian noise (AWGN) is addressed. Because a reciprocal process not always admits a well-behaved first-order causal innovations equivalent representation over the whole parameter space, in general the problems cannot be directly solved by standard recursive algorithms reported in the literature for Markov processes.

The n-variate zero-mean RGP \( \{x(k) \in \mathbb{R}^n, \text{Mdn} \} \) is described by the usual noncausal model [1]

\[
M_0(k)x(k) + \sum_{m=1}^{k-1} M_m(k)x(m-1) = e(k), \quad M+1 \text{is PSD} \quad (1)
\]

where the conjugate sequence \( \{e(k) \in \mathbb{R}^n, \text{Mdn} \} \) is zero-mean and Gaussian. The joint probability measure of the boundary r.v.'s \( \{X(M), X(N)\} \) is also given. The observed sequence \( \{z(k) \in \mathbb{R}^n \} \) is assumed as

\[
z(k) = A(k)x(k) + w(k), \quad M \leq k \leq N. \quad (2)
\]

where \( \{w(k) \in \mathbb{R}^n \} \) is the AWGN sequence, with known covariance matrix.

II. FILTER AND SMOOTHERS

The innovations technique has been applied to derive in an unified framework the MMSE Kalman-like filter and smoothers of \( \{x(k) \} \) based on the observation \( \{z(k) \} \). A finite set of recursive equations relating the filtered sequence \( \{\hat{x}(k)\} \), the one-step predicted sequence \( \{\hat{x}(k|k-1)\} \), the two-step predicted sequence \( \{\hat{x}(k|k-1)\} \) and the corresponding covariance error matrices (giving the MSE estimator performance) have been obtained. In particular, the filter equation is

\[
x(k|k) = x(k|k-1) + G(k) [z(k) - A(k)x(k|k-1)], \quad M \leq k \leq N, \quad (3)
\]

where

\[
x(k|k) = M_{0}^{-1}(k)[M_{0}(k)x(k|k-1) + M_{1}(k)x(k+1|k-1)], \quad M+1 \leq k \leq N \quad (4)
\]

and \( \{G(k)\} \) is the filter gains sequence, recursively computed from the second-order statistics of the estimator. A peculiar aspect of the above solution is that in Eq.(4) the two-step predicted sequence \( \{\hat{x}(k|k+1)\} \) is directly involved; this is a straightforward of the noncausal structure of the model in Eq.(1).

The general (recursive) solution of the estimation problem can be obtained by exploiting some properties of the RGP's deduced in [4,6] by introducing a complete class of 'tapped-delayed-line' circuit models for the (recursive) generation of the RGP's. The assigned boundary conditions for the r.v.'s \( \{X(M), X(N)\} \) specify the related circuit model so that the filter parameters in Eq.(2) can be recursively computed. An example of the performance of the proposed filter is reported in Fig. 1 for a case of Dirichlet boundary conditions. When the model in Eq.(1) constitutes the reciprocal representation of a Markov process the above filter equations reduce to those of the usual Kalman filter [6].

It can be shown that the filtered estimates constitute a sufficient statistic with respect to the observations \( \{z(k)\} \) for the smoothing estimates \( \{\hat{x}(k)\} \), so that the general expression of the proposed smoothers is

\[
x(k|k) = x(k|k)+ \sum_{m=k+1}^{N} P(k|m) \left( x(m|m) - (A(m-1)x(m-1|m-1)) \right), \quad (5)
\]

where the coefficients \( P(k|m) \) and \( A(m-1) \) are computed on the basis of the second-order smoother statistics. From Eq.(5) recursive expressions for the fixed-point, fixed-interval and fixed-lag smoothers (with the related MSE performance) can be directly obtained.

The particular case of fixed-interval smoothing is considered in [1,3] following a different approach. In [5] the estimation problem has been solved for the particular case of pinned-to-zero RGP's.

REFERENCES


Filler MSE of an univariate RGP with \( M=1, N=15, M_0 = 0.67, M_2 = 0.33 \in (1) \) and \( A(k-1) \in (2) \); the boundary r.v.'s are zero-mean jointly Gaussian with \( E[X(M)X(N)]=11.2, E[X^2(M)]=2.84, E[X(M)X(N)]=0.74 \); the noise variance is \( 20 \) (a), \( 10 \) (b), 5 (c), 1 (d). Simulation results based on 1000 independent trials are indicated by asterisks (**).