Model Parameter Estimation for Reciprocal Gaussian Random Processes
R. Cusani, E. Baccarelli, and G. Di Blasio

Abstract—The problem of estimating the model parameters of a discrete-index reciprocal Gaussian random process from a limited number of noisy observations is addressed. The general case of a first-order multivariate process is analyzed, stating its basic properties and deriving a linear equation set that relates the model parameters (including the unknown variance of the observation noise) to the (generally nonstationary) autocorrelation function of the observed process. It generalizes to the reciprocal processes the so-called 'high-order Yule-Walker equations' for AR processes. Based on these results, a practical estimation algorithm is proposed.

I. INTRODUCTION

Reciprocal Gaussian random processes (RGP's), also known as noncausal Gauss-Markov random processes or 1-D noncausal Gauss-Markov random fields, have received increased attention in the last years. They are well suited to describe physical models where a certain degree of noncausality is present such as those encountered in some control problems [7], in the mapping of spatial measurements, or in image processing [2], [5].

In this correspondence, the problem of estimating the parameters of an homogeneous discrete-index multivariate RGP in additive white noise with unknown covariance matrix is addressed. A general (matrix) relationship between the model parameters (including the covariance of the observation noise) and the nonstationary autocorrelation function (ACF) of the observed process is found; from this, a nonhomogeneous linear equation system is then built-up involving some samples of the observed ACF. Its solution gives simple expressions for the model parameters as a function of the observed ACF, from which a simple estimation algorithm is directly derived. When the observations are noise-free, alternative expressions for the model parameters are obtained involving short-term samples of the observed ACF.

An alternative procedure for the identification of homogeneous noncausal Gauss-Markov random fields was proposed in [9] (and in some related papers) with explicit reference to the 2-D case. Basically, the maximum likelihood estimates of the model parameters are obtained from the available observations (eventually affected by additive noise) through an iterative algorithm minimizing the so-called 'negative likelihood function.' This procedure (and its related 1-D version) exhibits a computational complexity proportional to the number of available data samples and requires that the noise variance is known, or is separately estimated. On the other hand, the

solution described in the present correspondence does not involve time-consuming iterative search algorithms and its complexity is independent from the number of available data samples. Moreover, the variance of the observation noise is estimated together with the model parameters of the process.

II. THE SOURCE AND OBSERVATION MODELS

An n-variate discrete-index homogeneous zero-mean first-order RGP \( \{ x(k) \in \mathbb{R}^n \} \) is described by the second-order equation (see, for example, [1], [5])

\[
    x(k) = \Phi_x x(k-1) + \Phi_u x(k+1) + u(k),
\]

\[
    1 \leq k \leq (N-1).
\]

(1)

The 'reciprocal' innovations process \( \{ u(k) \in \mathbb{R}^n \} \), generally known as 'conjugate process' [1], is assumed stationary zero-mean Gaussian with assigned covariance matrix \( \Sigma_u \), and is biorthogonal to \( \{ x(k) \} \), that is \( x(r) \perp u(s) \) for \( r \neq s \). Many kinds of boundary conditions (e.g., 'Dirichlet' or 'cyclic') can be associated to the above model [1].

The process described by (1) and by the assigned boundary conditions in general does not admit a well-behaved finite-order autoregressive (or causal) representation over the whole index space \([0, N]\), including both endpoints, giving a Markov version (in the stochastic sense) of the assigned process. As a consequence of this, it can be shown [3] that for the source model in (1) a 'valid parameter space' (in the sense of [8] and [9]) is obtained if and only if the matrices \( \Phi_x, \Phi_u, K_u \) satisfy the following conditions:

1. Reciprocity: The second-order difference operator associated to the model in (1) is self-adjoint, that is, \( \Phi_x = \Phi_x^T \).
2. Normality: The operator \( \Phi_x \) is 'normal' in the sense of Chap. V in [6], that is, \( \Phi_x \Phi_x^T = \Phi_x^T \Phi_x \).
3. Consistency: The operator \( \Phi_x \) and the covariance matrix \( \Sigma_u \) commute, that is, \( K_u \Phi_x = \Phi_x K_u \).
4. Positivity of the power spectral density matrix: The eigenvalues \( \{ \lambda_i, i = 1, \ldots, n \} \) of the matrix \( \Phi_x \) satisfy the inequalities \( 0 < |\lambda_i| < 0.5 \), \( i = 1, \ldots, n \).

In particular, it is shown in [3] and [10] that condition 1 is a consequence of a reciprocal property of the process defined by (1). Formally analogous conditions are also reported in Section 3 of [1]. Condition 4 guarantees that the power spectral density matrix of \( \{ u(k) \} \) is positive everywhere. Similar conditions have been assumed in Section V of [8] and Section V of [9], referring to the identification problem of a 'valid parameter space' for the 'source' process \( \{ x(k) \} \). The conditions 2), 3) (derived in [3] and [10]) are due to the fact that the power spectral density matrix of the process \( \{ u(k) \} \) is hermitian; as a consequence it must be diagonalized by a (matrix) unitary operator.

From the above conditions the model in (1) becomes

\[
    x(k) = \Phi_x x(k-1) + \Phi_u x(k+1) + u(k),
\]

\[
    1 \leq k \leq (N-1)
\]

(2)

and from the biorthogonality property of the RGP's we have that

\[
    E\left\{ u(k)u^T(k-m) \right\} =
    \begin{cases} 
        K_u & \text{for } m = 0, \\
        -K_u \Phi_x & \text{for } m = \pm 1, \\
        -K_u \Phi_u K_u & \text{for } m = \pm 1, \\
        0 & \text{for } |m| \geq 2.
    \end{cases}
\]

(3)
It is also assumed that the process \(\{x(k)\}\) is corrupted by a stationary zero-mean additive white noise process \(\{w(k)\} \in \mathbb{R}^M, 0 \leq k \leq N\) with a (positive definite) covariance matrix \(R_w\), so that the observed sequence \(\{y(k)\}\) is modeled as
\[
y(k) = x(k) + w(k), \quad 0 \leq k \leq N
\]
and its ACF \(R_y(k, m) = E[y(k)y^T(m)]\) can be expressed as
\[
R_y(k, m) = R_y(k, m) + R_w \delta(k, m), \quad R_y(k, m) = E[x(k)x^T(m)]
\]
and \(\delta(\cdot, \cdot)\) is the Kronecker delta.  

III. BASIC EQUATIONS FOR THE COMPUTATION OF THE MODEL PARAMETERS

Let us define \(\Phi_-, \Phi_+\) as the symmetric and the antisymmetric part of the matrix \(\Phi\), so that \(\Phi_- = \Phi_S + \Phi_A, \Phi_+ = \Phi_S - \Phi_A\). A general relationship between \(R_y(k, m)\) and the parameters \(\Phi_-, \Phi_+\). \(K_w\) is easily obtained from (2) and (3) as
\[
\begin{align*}
R_y(k, m, k - 1) &+ R_y(k - m, k + 1) \Phi_+ \\
&+ R_y(k, m + 1) \Phi_- = K_w(k - m, k).
\end{align*}
\]
Writing (5) for \(m = +1\) and \(m = -2\) and replacing \(R_y(k, m)\) by \(R_y(k, m)\), we have
\[
\begin{align*}
\{R_y(k, k - 2) + R_y(k - 2, k)\} \Phi_+ \\
+ \{R_y(k, k - 1) - R_y(k - 2, k)\} \Phi_- &+ R_y(k - 2, k) = 0.
\end{align*}
\]
Equations (6) and (7) constitute a nonhomogeneous linear algebraic system from which the (matrix) parameters \(\Phi_-, \Phi_+\) can be computed. For this purpose, let us define
\[
\begin{align*}
A(k) &= R_y(k - 2, k - 1) + R_y(k - 2, k), \\
B(k) &= R_y(k - 2, k + 1) - R_y(k - 2, k), \\
C(k) &= R_y(k - 2, k + 1) + R_y(k - 2, k + 1), \\
D(k) &= R_y(k - 2, k - 1) + R_y(k - 2, k - 1).
\end{align*}
\]
From the matrix inversion lemma for \(2 \times 2\) block matrices, after some manipulations, we obtain the solution
\[
\Phi_- = \left(A(k) - D(k)B(k)^{-1}C(k)^{-1}\right)^{-1} \\
\times \left\{R_y(k - 2, k) - D(k)B(k)^{-1}R_y(k - 2, k)\right\},
\]
\[
\Phi_+ = B(k)^{-1} \left[ -C(k)A(k) - D(k)B(k)^{-1}C(k)^{-1} \right] \\
\times \left\{R_y(k - 2, k) + [I + C(k)] \\
\times \left( A(k) - D(k)B(k)^{-1}C(k)^{-1} \right) \right\} \\
\times R_y(k - 2, k + 2)
\]
\[
\Phi_+ = B(k)^{-1} \left[ -C(k)A(k) - D(k)B(k)^{-1}C(k)^{-1} \right] \\
\times \left\{R_y(k - 2, k) + [I + C(k)] \\
\times \left( A(k) - D(k)B(k)^{-1}C(k)^{-1} \right) \right\} \\
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\times R_y(k - 2, k + 2)
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\Phi_+ = B(k)^{-1} \left[ -C(k)A(k) - D(k)B(k)^{-1}C(k)^{-1} \right] \\
\times \left\{R_y(k - 2, k) + [I + C(k)] \\
\times \left( A(k) - D(k)B(k)^{-1}C(k)^{-1} \right) \right\} \\
\times R_y(k - 2, k + 2)
\]

where \(\Phi_-, \Phi_+\) are known from (9) and (10). From the above expression, it is observed that \(R_w\) depends nonlinearly on \(\Phi_-, \Phi_+\) but it is a linear function of the ACF \(R_y(k, m)\). Finally, writing (5) for \(m = 0\), the parameter \(K_w\) is found as
\[
K_w = R_y(k, k) - R_w + R_y(k - 1, k - 1)(\Phi_+ - \Phi_-)
\]
\[
- R_y(k + 1, k - 1)(\Phi_+ + \Phi_-)
\]
where \(\Phi_-, \Phi_+, R_w\) have been preliminarily computed from (9)-(11).

IV. THE ESTIMATION ALGORITHM

From (9)-(12), the model parameters \(\Phi_-, \Phi_+, R_w, K_w\) can be computed if the ACF \(\{R_y(r, s)\}\) of the observation process is known for \(r = k - 2, \ldots, k + 2\) and \(s = k - 1, k + 1\), where \(k\) is any value in the interval \([3, N - 3]\). In practice, however, the ACF must be estimated from one (or more) realizations of the process \(\{y(k)\}\).


**TABLE I**

<table>
<thead>
<tr>
<th>Case of noisy observations</th>
<th>$R_u(k-2,k-1)$</th>
<th>$R_u(k-2,k+1)$</th>
<th>$R_u(k+2,k-1)$</th>
<th>$R_u(k+2,k+1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Noisy case</td>
<td>$R_u(k-1,k-1)$</td>
<td>$R_u(k-1,k+1)$</td>
<td>$R_u(k+1,k-1)$</td>
<td>$R_u(k+1,k+1)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case of noisy observations</th>
<th>$R_u(k+2,k)$</th>
<th>$R_u(k,k+1)$</th>
<th>$R_u(k,k+1)$</th>
<th>$R_u(k,k)$</th>
<th>$R_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Noisy case</td>
<td>$R_u(k+1,k)$</td>
<td>$R_u(k,k-1)$</td>
<td>$R_u(k,k-1)$</td>
<td>$R_u(k,k)$</td>
<td>0</td>
</tr>
</tbody>
</table>

![Fig. 1](image1.png)

**Fig. 1.** Behavior of the ACF: $R_u(k,k)$ (---), $R_u(k,k+1)$ (---). $R_u(k,k+2)$ (----). $R_u(k,k+3)$ (-----) as a function of $k$, estimated as in (18) from $NR = 10^4$ independent runs. A univariate ($n = 1$) RGP of length $N + 1 = 21$ samples with pinned-to-zero boundary conditions and model parameters $\Phi_S = 0.5, K_u = 1.5, R_u = 5$ has been assumed.

Whilst $\{R_u(r,s)\}$ is not stationary, in general, it cannot be exactly computed by averaging dot products from a unique realization, as it is usually done in the stationary case.

Let us assume that NR realizations $\{y^{(j)}(k)\}$, $0 \leq k \leq N$, $j = 1,\ldots, NR$ of the process $\{y(k)\}$ are available. In this case, $\{R_u(r,s)\}$ can be simply estimated as

$$\hat{R}_u(r,s) = \frac{1}{NR} \sum_{j=1}^{NR} y^{(j)}(r)s^{(j)}(s), \quad 2 \leq r \leq k + 2,$$

$$k - 1 \leq s \leq k + 1, \quad 3 \leq k \leq N - 3. \quad (18)$$

As an example, in Figs. 1 and 2 the behavior of some estimates of $\{R_u(r,s)\}$ obtained from simulation results by averaging over $NR = 10^4$ independent runs are reported. The observation noise is assumed to be Gaussian. Two univariate processes with different model parameters and pinned-to-zero boundary conditions (i.e., $x(0) = x(N) = 0$) have been considered. This constitutes a particular case of the more general Dirichlet boundary conditions, often occurring in many practical situations (see, for example, [7]).

In the simulations, we have adopted the recursive procedure described in the Appendix to generate the univariate RGP sequence. It is worthwhile to note that the sequence $\{x(k)\}$ in Fig. 1 is highly correlated, so that the behavior of the ACF is strongly influenced by the boundary conditions. On the other hand, the sequence $\{y(k)\}$ in Fig. 2 is less correlated, and its ACF is nearly constant for values of $k$ far from 0 and $N$.

The estimates $\hat{\Phi}_S^{(k)}$, $\hat{\Phi}_A^{(k)}$ of $\Phi_S$, $\Phi_A$ can be now computed (for $3 \leq k \leq (N-3)$) as in (9) and (10) by replacing the true ACF

![Fig. 2](image2.png)

**Fig. 2.** Same as in Fig. 1, with model parameters $\Phi_S = 0.3, K_u = 0.3, R_u = 0.2$.

![Fig. 3](image3.png)

**Fig. 3.** Behavior of $\hat{\Phi}_S^{(k)}$ (---), $\hat{K}_u^{(k)}$ (---). $\hat{R}_u^{(k)}$ (-----) for the same process of Fig. 1, computed from the estimated ACF of Fig. 1.

**TABLE II**

<table>
<thead>
<tr>
<th>$N+1=21$</th>
<th>$\Phi_S = 0.5, K_u = 1.5, R_u = 5$</th>
<th>$\Phi_S = 0.3, K_u = 0.3, R_u = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\Phi}_S$</td>
<td>0.4993</td>
<td>0.30236</td>
</tr>
<tr>
<td>$\hat{K}_u$</td>
<td>1.5528</td>
<td>0.29263</td>
</tr>
<tr>
<td>$\hat{R}_u$</td>
<td>5.0137</td>
<td>0.20360</td>
</tr>
</tbody>
</table>

![Estimated models parameters](image4.png)

**Estimated models parameters**

In Figs. 3 and 4, the behavior of $\hat{\Phi}_S^{(k)}$, $\hat{K}_u^{(k)}$, $\hat{R}_u^{(k)}$ obtained from simulation results is shown. In Table II, the corresponding estimates $\hat{\Phi}_S$, $\hat{K}_u$, $\hat{R}_u$ are reported.

We observe that in the noisless case if the ACF $\{R_u(r,s)\}$ is positive definite for $0 < r < N$ and $0 < s < N$, then the algebraic system (6) and (7) is generally guaranteed to be nonsingular for $1 \leq k \leq N - 1$. In the case of noisy observations, however, the definite
positiveness of \( \{ R_s(r, s) \} \) and, as a consequence, of \( \{ R_s(r, s) \} \) too not necessarily ensures the same property. A similar situation is also encountered in AR spectral estimation [4].

V. CONCLUSION

The procedure proposed in this correspondence allows computation of the model parameters of a first-order multivariate discrete-index homogeneous RGP described by the noncausal model of (1) on the basis of measurements of the (generally nonstationary) ACF of the observation process only.

Further research is currently in progress. In particular, the robustness of the proposed estimators with respect to the number of available realizations NR and to their length \( N \) is being evaluated for assigned values of the model parameters. Moreover, starting from the same basic equations (9)-(12), alternative strategies for combining the ACF estimates obtained at different index steps could be devised, depending on the assumed boundary situations and on the available \textit{a priori} knowledge about the ACF behavior of the source process. Finally, the proposed approach could be extended to the identification of 2-D noncausal Gauss-Markov random fields.

APPENDIX

RECURSIVE GENERATION OF AN UNIVARIATE PINNED-TO-ZERO RGP

A sample sequence \( \{ x(k) \} \) of a univariate RGP with pinned-to-zero boundary conditions can be generated starting from a stationary zero-mean white Gaussian sequence \( \{ v(k) \in \mathbb{R}^*, 0 \leq k \leq N-1 \} \) with variance \( Q \).

The 'reciprocal' innovation process \( \{ u(k) \in \mathbb{R}^*, 1 \leq k \leq N-1 \} \) is then obtained as \( u(k) = v(k) - v(k) = v(k) = v(k-1) \), with \( 1 \leq k \leq (N-1) \). It is easily verified that the sequence \( \{ u(k) \} \) satisfies (3) with \( \Phi_s = \alpha/(1 + \alpha^2) \) and \( K_s = Q/(1 + \alpha^2) \). The sequence \( \{ x(k) \in \mathbb{R}^*, 0 \leq k \leq N \} \) is then directly given by (see, for example, Section III of [1])

\[
x(k) = \sum_{s=1}^{N-1} G(k, s) u(s)
= \sum_{s=1}^{N-1} 2Q \frac{N-1}{N} \sin(\pi/N) \sin(k \pi/N) \frac{1}{1 + \alpha^2 - 2 \alpha \cos(\pi/N)} u(s), \quad 1 \leq k \leq (N-1)
\]

with \( x(0) = x(N) = 0 \), where \( \{ G(k, s) \in \mathbb{R}^*, 0 \leq k \leq N, 0 \leq s \leq N \} \) is the Green function associated to the model of (2) (see Chap. IV of [3]).

Some general remarks about the basic properties of the Green function have been pointed out for example in [1] and in Chap. IV of [3]. Alternative methods for the recursive generation of univariate reciprocal sequences (substantially based on the Cholesky decomposition of the so-called 'potential matrix' of the RGP) have been given in Section V of [1] and in Section VI of [2].

REFERENCES