where \( \mathbf{v}_{k+1} \) is the right singular vector of \( \mathbf{G}(k_1, k_2) \) associated with the \((L+1)^{th}\) singular value.

The choice of \( k_1 \) and \( k_2 \) may affect the performance of the algorithm. However, it is not difficult to combine the cyclic statistics for some or all (distinct) \( k_1 \)'s and \( k_2 \)'s. In doing so, the criterion in (51) can be modified by replacing \( \mathbf{G}(k_1, k_2) \) by

\[
\mathbf{G} = \left[ \mathbf{G}'(0, 1), \ldots, \mathbf{G}'(0, T - 1), \mathbf{G}'(1, 2), \ldots, \mathbf{G}'(1, T - 1), \ldots, \mathbf{G}'(T - 2, T - 1) \right].
\]

(53)

Such modification may improve the performance.

V. CONCLUDING REMARKS

We established several different necessary and sufficient conditions for the identifiability of a possibly nonminimum phase channel from its output cyclic autocorrelation functions. In comparison to the time-domain approach presented earlier in [12], the frequency-domain approach to the channel identification problem gives more insight into the issue of channel identifiability. It also provides the basis for new channel identification algorithms.

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Recursive Filtering and Smoothing for Reciprocal Gaussian Processes—Pinned Boundary Case

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Abstract—The least square estimation problem for pinned-to-zero discrete-index reciprocal Gaussian processes in additive white noise is solved, thus completing and extending some previous results available in the literature. In particular, following the innovations approach a (finite) set of recursive equations is obtained for the filter and for the three standard classes of smoothers (fixed-point, fixed-interval, fixed-lag). Recursive expressions for the mean square performance of the proposed estimators are also given.

Index Terms—Reciprocal processes, innovations method, recursive estimators.

I. INTRODUCTION

A discrete-index multivariate real reciprocal Gaussian random process \( \{ x(k) \in \mathbb{R}^n, M \leq k \leq N \} \) is defined on an assigned probability space \((\Omega, \mathcal{A}, P)\) by the well-known self-adjoint second-order noncausal difference model [1]

\[
\begin{align*}
M(k)x(k) - M(k)x(k+1) - M(k-1)x(k-1) &= \varepsilon(k), \\
M + 1 \leq k \leq N - 1
\end{align*}
\]

(1)

where \( \{ M(k) \} \) are deterministic sequences of \( n \times n \) matrices. The conjugate process

\[
\{ \varepsilon(k) \in \mathbb{R}^n, (M+1) \leq k \leq (N-1) \}
\]

is bi-orthogonal to \( \{ x(k) \} \), that is,

\[
E[ \varepsilon(k)^T s(s) ] = I(k, s),
\]

and its covariance matrix is given by [1, eqs. (3.4a), (3.4b)]. Boundary random values with singular probability measures (pinned-to-zero) are considered in this paper, i.e., \( x(M) = x(N) = 0, P-\text{a.s.} \)

It is assumed that the sequence \( \{ x(k) \} \) is amplitude-modulated and then sequentially transmitted through a noisy communication channel for increasing values of the index \( k \). As a consequence, the observed sequence \( \{ y(k) \in \mathbb{R}^n \} \) is modeled as

\[
\begin{align*}
y(k) &= G(k)x(k) + w(k), \\
M + 1 \leq k \leq N - 1
\end{align*}
\]

(2)

where \( \{ w(k) \in \mathbb{R}^n \} \) is an additive white Gaussian noise (AWGN) process, independent of \( \{ x(k) \} \), with covariance matrices \( \{ R(k) \} = \{ I(k) \} \) are known and uniformly limited \( n \times n \) matrices. It is also assumed that the matrices \( \{ M_0(k) \} \) and \( \{ R(k) \} \) are nonsingular.

In this correspondence, the problem of estimating a pinned-to-zero RGP in AWGN is addressed. Pinned-to-zero boundary conditions represent an important subclass of the more general Dirichlet boundary conditions [1]. In this case, an RGP does not always admit a well-behaved first-order causal white innovations representation over the whole parameter space (e.g., including both ending points) giving a Markov version (in the stochastic sense) of the assigned process. As a consequence, the pertaining estimation problems cannot be directly

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solved by standard recursive algorithms which are reported in the literature for Gauss–Markov processes (GMP’s) [2]. Furthermore, the solutions of the estimation problems derived for pinned-to-zero boundary conditions could be extended to solve the corresponding problems for general Dirichlet boundary conditions. Some preliminary results about these more general problems are also reported in [5].

The solution for the fixed-interval smoothing (often considered as the natural problem for RGP’s) has been given in [1], [3]. However, to the authors’ knowledge no explicit formulas are available for its performance neither for the filter nor for the fixed-point and fixed-lag smoothers. Regarding this point, we observe that in some real-time applications the delay associated to the fixed-interval smoother cannot be tolerated so that the filter may constitute an effective alternative if the resulting accuracy loss is small enough.

The aim of this correspondence is then to give recursive expressions for the filter, the smoothers, and their mean square performance in a unified approach based on the innovations theory.

II. THE FILTER

In order to derive the minimum mean square error (MMSE) filtering formulas for the pinned RGP’s, let us denote by \( Y^k = \{ y(M+1), \cdots, y(k) \} \) the observed sequence until step \( k \); by \( F_k = \{ \{ Y^k \} \} \), \( k \geq M + 1 \), the \( \sigma \)-algebra generated by \( Y^k \) (with \( F_0 = \{ \{ \phi \} \}) \); by \( \hat{x}(k \mid k-m) = \mathbb{E}\{ X(k) \mid F_{k-m} \} \), the filtered sequence (for \( m = 0 \)) and the one-step predictions sequence (for \( m = 0, 1 \)); by \( S(k \mid k-m) \), with \( m = 0, 1 \), their corresponding error covariance matrices.

On the basis of the observation model in (2) and using the well-known Martingale Difference representation theorem, the filter equation is given by

\[
\hat{x}(k \mid k) = \hat{x}(k \mid k-1) + G(k)[y(k) - \Gamma(k)\hat{x}(k \mid k-1)],
\]

\( M+1 \leq k \leq N-1 \). (3)

From the assumptions of Section I it follows that the gains sequence \( \{ G(k) \} \) in (3) is necessarily \( F_k \)-predictable, so that it can be expressed as

\[
G(k) = [S(k \mid k-1)]^T[k][F(k)S(k \mid k-1)]^T + R(k)^{-1},
\]

\( M+1 \leq k \leq N-1 \). (4)

Starting from the integral form of the difference model in (1) (reported, for example, in [1, eq. (3.13)]) and on the basis of (2), after some algebra the one-step predictions sequence in (3) can be recursively calculated as

\[
\hat{x}(k \mid k-1) = M_{0}^{-1}(k)[M_{0}^{T}(k-1) + M_{0}(k)H(k-1, N)]^{-1} \cdot \hat{x}(k-1 \mid k-1), \quad M+1 \leq k \leq N-1
\]

where

\[
H(k-1, N) = G(k+1, k-1, N)M_{0}^{T}(k-1), \quad M+1 \leq k \leq N-1
\]

and \( \{ G(r, s; k-1, N), (k-1) \leq r \leq N, k \leq s \leq N-1 \} \) is the Green’s function associated to (1) over the subinterval \( [k-1, N] \) of the whole assigned index-space \([M, N]\) (it is defined here according to [1, eq. (3.10)]. It is worthwhile observing that the recursive computation of the one-step prediction sequence in (5) requires the use of the integral form of the model in (1) (and then, the use of the Green’s functions of (6)), so that the effect of the assigned boundary values is taken into account in the solution of the problem. This is a direct consequence of the noncausal structure of the model in (1).

In order to express the gains sequence of (4) in a recursive form, first of all the second-order statistics of the estimator must be calculated. Regarding the covariance matrix of the one-step prediction error, from (1)–(5) and from the bi-orthogonality property mentioned in Section I we obtain

\[
S(k \mid k-1) = M_{0}^{-1}(k) + M_{0}^{-1}(k)[C(k)S(k \mid k-1)C^{T}(k) + M_{0}(k)]G(k+1, k+1; k-1, N)M_{0}^{T}(k) \cdot [I - \delta(k, N-1)] + 2V_0(k, k-1; N)]M_{0}^{-1}(k), \quad M+1 \leq k \leq N-1
\]

with

\[
C(k) = M_{0}^{T}(k-1) + M_{0}(k)H(k-1, N), \quad M+1 \leq k \leq N-1
\]

and where \( V_0(k, k-1; N) \) is the symmetric part of the matrix \( M_{0}^{T}(k-1)S(k \mid k-1)H^{T}(k-1, N)M_{0}^{T}(k) \).
Finally, on the basis of (2)-(4), the mean square filter performance is given by

\[
S(k \mid k) = (I - G(k) \Gamma(k)) S(k \mid k - 1).
\]

\[
M + 1 \leq k \leq N - 1. \quad (9)
\]

The equations of this section, initialized with zero values (i.e., \(\hat{x}(M \mid M) = 0, S(M \mid M) = 0\)), allows the recursive evaluation of the desired filtered estimates. More details about their derivations can be found in [4], [7].

The proposed filter in (3) is finite-dimensional and linear. As a direct consequence of the followed approach, it exhibits a recursive structure formally identical to the classic Kalman filter for GMP’s and its implementation requires an equal real-time computational effort. In fact, the Green’s functions do not depend on the observed sequence, so that they can be precomputed off-line on the basis of the model in (1) only. Furthermore, for any subinterval \([k - 1, N]\) of the index space only two samples of the Green’s function (those appearing in (6) and (7)) must be computed, and this can be carried out by means of computationally efficient (and numerically stable) algorithms proposed in the literature for the solution of block-tridiagonal systems.

Remark 1: The filtering formulas of this section do not change when \(X(M)\) is a random variable with nonsingular Gaussian distribution and \(x(N) = 0\). Of course, in this case the initial conditions \(\{\hat{x}(M \mid M); S(M \mid M)\}\) must be calculated on the basis of the available information (i.e., the a priori statistics of \(X(M)\) and the initial observation \(y(M)\), generally noisy).

III. THE SMOOTHERS

Recursive solutions for the fixed-point, fixed-interval, and fixed-lag smoothing problems can be derived on the basis of the available filtered estimates only. First of all, let us consider the following general smoothing formulas, easily deduced starting from the observation model in (2) by applying the innovations method [6]

\[
\dot{x}(k \mid b) = \dot{x}(k \mid k) + \sum_{m=k+1}^{b} S(k; m) \Gamma(k) (m)
\]

\[
\cdot [\Gamma(m) S(m \mid m - 1) \Gamma^T(m) + R(m)]^{-1}
\]

\[
\cdot (g(m) - \Gamma(m) \hat{x}(m \mid m - 1)), \quad M + 1 \leq k \leq b
\]

\[
S(k \mid b) = S(k \mid k) - \sum_{m=k+1}^{b} S(k; m) \Gamma(m)
\]

\[
\cdot [\Gamma(m) S(m \mid m - 1) \Gamma^T(m) + R(m)]^{-1}
\]

\[
\cdot \Gamma(m) S^T(k; m), \quad M + 1 \leq k \leq b
\]

where \(b\) is an assigned index-value (i.e., \(k < b \leq N - 1\)), \(\hat{x}(k \mid b)\) is the smoothed estimate, \(S(k \mid b)\) its corresponding error covariance matrix, and

\[
S(k; m) = E[(x(k) - \hat{x}(k \mid k - 1))(x(m) - \hat{x}(m \mid m - 1))],
\]

\[k + 1 < m \leq b\]

are one-step prediction error covariance matrices. From the above expressions and on the basis of the bi-orthogonality property of Section I the latter can be expressed as

\[
S(k; m) = S(k \mid k-1) \Phi(k;m-1), \quad m \geq k + 1
\]

with

\[
S(k; k) = S(k \mid k - 1)
\]

where the state-transition matrices \(\Phi(k; m - 1)\) are recursively calculated as

\[
\Phi(k; m - 1) = \Phi(k; m - 2) [I - \Gamma^T(m - 1) \Gamma(m - 1)]
\]

\[
\cdot [M_x(m - 1) + H^T(m - 1, N)]
\]

\[
\cdot M^T_x(m)[M_0(m)]^{-1}, \quad m \geq k + 1
\]

with

\[
\Phi(k; k - 1) = I.
\]

Finally, from (3)-(5) the general smoothing formula in (10) can be rewritten as

\[
\dot{x}(k \mid b) = \dot{x}(k \mid k) + \sum_{m=k+1}^{b} S(k; m) S(m \mid m - 1)^{-1}
\]

\[
\cdot \{ \dot{x}(m \mid m) - \beta^T(m - 1) \dot{x}(m - 1 \mid m - 1) \}
\]

\[M + 1 \leq k \leq b\]

where the sequence \(\{\beta(m)\}\) is defined as

\[
\beta(m) = [M_x(m) + H^T(m, N) M^T_x(m + 1)] [M_0(m + 1)]^{-1}.
\]
A. Fixed-Point Smoothing

Directly from (11)–(14) the following recursive formulas are derived:

\[
\hat{x}(k \mid b + 1) = \hat{x}(k \mid b) + S(k; b + 1)[S(b + 1 \mid b)]^{-1} \cdot \hat{x}(b + 1 \mid b + 1) - \beta^T(k)\hat{y}(b \mid b),
\]

\[b \geq k, \quad k \text{ fixed} \quad (16)\]

\[
S(k \mid b + 1) = S(k \mid b) - S(k; b + 1)\Gamma^T(b + 1)[\Gamma(b + 1) - \beta^T(k)S(b + 1 \mid b + 1) + \Gamma(b + 1)]^{-1} \cdot \Gamma(b + 1)S^T(k; b + 1), \quad b \geq k, \quad k \text{ fixed} \quad (17)
\]

with

\[
S(k; b + 1) = S(k; b)[I - \Gamma^T(b)G^T(b)]\beta(b),
\]

\[b \geq k, \quad k \text{ fixed} \quad (18)\]

The recursions in (16)–(18) must be initialised by \(\hat{x}(k \mid k)\), \(S(k \mid k)\), and \(S(k \mid k - 1)\), respectively.

B. Fixed-Interval Smoothing

On the basis of (12) and (13), from (11)–(14) and after some algebra the following backward recursions can be obtained for the fixed-interval smoother and for the corresponding error covariance matrix:

\[
\hat{x}(k - 1 \mid b) = \hat{x}(k - 1 \mid k - 1) + S(k - 1 \mid k - 1)\beta(k - 1)
\]

\[
\cdot [S(k - 1 \mid k - 1)]^{-1} \cdot \hat{x}(k \mid b) - \beta^T(k - 1)\hat{y}(k \mid b),
\]

\[b \geq k \quad (19)\]

\[
S(k - 1 \mid b) = S(k - 1 \mid k - 1) - S(k - 1 \mid k - 1)\beta(k - 1)
\]

\[
\cdot [S(k - 1 \mid k - 1)]^{-1} \cdot [S(k \mid b) - S(k \mid b)] - \beta^T(k - 1)S(k - 1 \mid b - 1),
\]

\[b \geq k \quad (20)\]

The above equations, initialised by \(\hat{x}(b \mid b)\) and \(S(b \mid b)\), allow to recursively compute the fixed-interval smoothed estimates on the basis of the available filtered estimates only.

Remark 2: Let us consider now the case considered in [1], [3] of Dirichlet boundary conditions and noise-free observations at the boundary points of the index space, i.e., \(y(M) = \pi(M), y(N) = \pi(N)\). Starting from the integral form of the difference model in (1), the fixed-interval smoothed sequence \(\hat{x}(k \mid N)\) in this case can be expressed as

\[
\hat{x}(k \mid N) = \hat{x}(k \mid N - 1) + G(k, M + 1; M, N)M^T_x(M)\pi(M)
\]

\[\cdot G(k, N - 1; M, N)M_x(N - 1)\pi(N),
\]

\[M + 1 \leq k \leq N - 1 \quad (21)\]

with \(\hat{x}(M \mid N) = y(M), \hat{x}(N \mid N) = y(N)\). The corresponding error covariance matrix sequence \(\hat{S}(k \mid N) = S(k \mid N - 1), M + 1 \leq k \leq N - 1, s(k \mid N) = \hat{S}(N \mid N) = 0\). In the above formulas \{\hat{x}(k \mid N - 1)\} and \{\hat{S}(k \mid N - 1)\} are, respectively, the fixed-interval smoothed sequence and its error covariance matrix sequence for the case of pinned-to-zero RGP's, computed as in (19) and (20) with \(b = N - 1\).

Remark 3: The implementations of the fixed-interval smoother proposed in [1], [3] extend to the RGP's the well-known Rauch–Tung–Striebel and the Mayne–Fraser formulas for GMP's. For the pinned boundary case, they do not require the explicit computation of the Green's functions of (6) and (7), so that they present some computational advantages over the smoother in (19).

C. Fixed-Lag Smoothing

In this case, from (11)–(14) the following forward recursion can be derived:

\[
\hat{x}(k + 1 \mid k + 1 + \Delta) = \beta^T(k)\hat{x}(k \mid k + S(k + 1 \mid k)
\]

\[
\cdot [S(k \mid k)\beta(k)]^{-1} [\hat{x}(k \mid k) - \hat{x}(k \mid k)]
\]

\[
+ \Phi(k + 1; k + \Delta)[S(k + 1 + \Delta \mid k + \Delta)]^{-1} \cdot [\hat{x}(k + 1 + \Delta \mid k + 1 + \Delta) - \beta^T(k + \Delta)
\]

\[
\cdot \hat{x}(k + \Delta \mid k + \Delta)],
\]

\[M + 1 \leq k \leq N - 2 - \Delta \quad (22)\]

\[
S(k + 1 \mid k + 1 + \Delta) = S(k + 1 \mid k) - S(k \mid k + \Delta)
\]

\[
\cdot [S(k \mid k)\beta(k)]^{-1} [S(k \mid k) - S(k \mid k + \Delta)]
\]

\[
\cdot [\beta^T(k)S(k \mid k)]^{-1} + \Phi(k + 1; k + \Delta)
\]

\[
\cdot [S(k + 1 + \Delta \mid k + \Delta)]^{-1} \cdot G(k + 1 + \Delta)\Gamma(k + 1 + \Delta)
\]

\[\cdot \Phi^T(k + 1; k + \Delta)S(k \mid k + \Delta)\]

\[M + 1 \leq k \leq N - 2 - \Delta \quad (23)\]

The above formulas must be initialised by \(\hat{x}(M + 1 \mid M + 1 + \Delta)\) and \(S(M + 1 \mid M + 1 + \Delta)\), available from the fixed-point smoothed solutions in (16)–(17).

In Figs. 1, 2 the filtering and the fixed-interval smoothing MSE's are reported for two RGP's with different parameters and for some different channel noise levels. The full agreement between the theoretical curves and the simulation results is appreciated. From the reported plots the accuracy loss of the filter with respect to the fixed-interval smoother can be evaluated. It results negligible in the case of weakly correlated RGP's, such as that of Fig. 1, while it is larger for the highly correlated RGP of Fig. 2.

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